

A L^2 TO L^∞ APPROACH FOR THE LANDAU EQUATION

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ABSTRACT. Consider the Landau equation with Coulomb potential in a periodic box. We develop a new $L^2 \rightarrow L^\infty$ framework to construct global unique solutions near Maxwellian with small L^∞ norm. The first step is to establish global L^2 estimates with strong velocity weight and time decay, under the assumption of L^∞ bound, which is further controlled by such L^2 estimates via De Giorgi's method [10] and [19]. The second step is to employ estimates in S_p spaces to control velocity derivatives to ensure uniqueness, which is based on Holder estimates via De Giorgi's method [10], [11], and [19].

1. INTRODUCTION

We consider the following generalized Landau equation:

$$\partial_t F + v \cdot \nabla_x F = Q(F, F) = \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} \phi(v - v') [F(v') \nabla_v F(v) - F(v) \nabla_v F(v')] dv' \right\}, \quad (1.1)$$

$$F(0, x, v) = F_0(x, v),$$

where $F(t, x, v) \geq 0$ is the spatially periodic distribution function for particles at time $t \geq 0$, with spatial coordinates $x = (x_1, x_2, x_3) \in [-\pi, \pi]^3 = \mathbb{T}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The non-negative matrix ϕ is

$$\phi^{ij}(v) = \left\{ \delta_{i,j} - \frac{v_i v_j}{|v|^2} \right\} |v|^{-1}. \quad (1.2)$$

As in the Boltzmann equation, it is well-known that Maxwellians are steady states to (1.1). Let μ be a normalized Maxwellian

$$\mu(v) = e^{-|v|^2}, \quad (1.3)$$

and set

$$F(t, x, v) = \mu(v) + \mu^{1/2}(v) f(t, x, v). \quad (1.4)$$

Then the standard perturbation $f(t, x, v)$ to μ satisfies

$$f_t + v \cdot \partial_x f + Lf = \Gamma(f, f), \quad (1.5)$$

$$f(0, x, v) = f_0(x, v), \quad (1.6)$$

where f_0 is the initial data satisfying the conservation laws:

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f_0(x, v) \sqrt{\mu} = \int_{\mathbb{T}^3 \times \mathbb{R}^3} v_i f_0(x, v) \sqrt{\mu} = \int_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 f_0(x, v) \sqrt{\mu} = 0. \quad (1.7)$$

The linear operator L and the nonlinear part Γ are defined as

$$L = -A - K, \quad (1.8)$$

$$\begin{aligned} Af &:= \mu^{-1/2} \partial_i \left\{ \mu^{1/2} \sigma^{ij} [\partial_j f + v_j f] \right\} \\ &= \partial_i [\sigma^{ij} \partial_j f] - \sigma^{ij} v_i v_j f + \partial_i \sigma^i f, \end{aligned} \quad (1.9)$$

$$Kf := -\mu^{-1/2} \partial_i \left\{ \mu \left[\phi^{ij} * \left\{ \mu^{1/2} [\partial_j f + v_j f] \right\} \right] \right\}, \quad (1.10)$$

$$\begin{aligned} \Gamma[g, f] &:= \partial_i \left[\left\{ \phi^{ij} * [\mu^{1/2} g] \right\} \partial_j f \right] - \left\{ \phi^{ij} * [v_i \mu^{1/2} g] \right\} \partial_j f \\ &\quad - \partial_i \left[\left\{ \phi^{ij} * [\mu^{1/2} \partial_j g] \right\} f \right] + \left\{ \phi^{ij} * [v_i \mu^{1/2} \partial_j g] \right\} f, \end{aligned} \quad (1.11)$$

$$\sigma_u^{ij}(v) := \phi^{ij} * u = \int_{\mathbb{R}^3} \phi^{ij}(v - v') u(v') dv', \quad (1.12)$$

$$\sigma = \sigma_\mu, \quad \sigma^i = \sigma^{ij} v_j. \quad (1.13)$$

To get L^∞ estimates, we rearrange (1.5) as follows:

$$f_t + v \cdot \nabla_x f = \bar{A}_f f + \bar{K}_f f, \quad (1.14)$$

$$\begin{aligned} \bar{A}_g f &:= \partial_i \left[\left\{ \phi^{ij} * [\mu + \mu^{1/2} g] \right\} \partial_j f \right] \\ &\quad - \left\{ \phi^{ij} * [v_i \mu^{1/2} g] \right\} \partial_j f - \left\{ \phi^{ij} * [\mu^{1/2} \partial_j g] \right\} \partial_i f \\ &=: \nabla_v \cdot (\sigma_G \nabla_v f) + a_g \cdot \nabla_v f, \end{aligned} \quad (1.15)$$

$$\begin{aligned} \bar{K}_g f &:= Kf + \partial_i \sigma^i f - \sigma^{ij} v_i v_j f \\ &\quad - \partial_i \left\{ \phi^{ij} * [\mu^{1/2} \partial_j g] \right\} f + \left\{ \phi^{ij} * [v_i \mu^{1/2} \partial_j g] \right\} f. \end{aligned} \quad (1.16)$$

Define the weighed norm and weighted energy associated with (1.5):

$$w := (1 + |v|), \quad |f|_{p,\vartheta}^p := \int_{\mathbb{R}^3} w^{p\vartheta} f^p dv, \quad \|f\|_{p,\vartheta}^p := \int_{\mathbb{T}^3 \times \mathbb{R}^3} w^{p\vartheta} f^p dx dv. \quad (1.17)$$

$$|f|_{\sigma,\vartheta}^2 := \int_{\mathbb{R}^3} w^{2\vartheta} [\sigma^{ij} \partial_i f \partial_j f + \sigma^{ij} v_i v_j f^2] dv, \quad (1.18)$$

$$\|f\|_{\sigma,\vartheta}^2 := \iint_{\mathbb{T}^3 \times \mathbb{R}^3} w^{2\vartheta} [\sigma^{ij} \partial_i f \partial_j f + \sigma^{ij} v_i v_j f^2] dv dx, \quad (1.19)$$

$$|f|_{\infty,\vartheta} = \sup_{\mathbb{R}^3} w^\vartheta(v) f(v), \quad \|f\|_{\infty,\vartheta} = \sup_{\mathbb{T}^3 \times \mathbb{R}^3} w^\vartheta(v) f(x, v). \quad (1.20)$$

$$\begin{aligned} |f|_2 &:= |f|_{2,0}, \quad \|f\|_2 := \|f\|_{2,0}, \\ |f|_\sigma &:= |f|_{\sigma,0}, \quad \|f\|_\sigma := \|f\|_{\sigma,0}, \\ |f|_\infty &:= |f|_{\infty,0}, \quad \|f\|_\infty := \|f\|_{\infty,0}, \end{aligned}$$

$$\langle f, g \rangle := \int_{\mathbb{R}^3} f g dv, \quad (f, g) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} f g dx dv, \quad (1.21)$$

$$\langle f, g \rangle_\sigma := \int_{\mathbb{R}^3} [\sigma^{ij} \partial_i f \partial_j g + \sigma^{ij} v_i v_j f g] dv, \quad (1.22)$$

$$(f, g)_\sigma := \iint_{\mathbb{T}^3 \times \mathbb{R}^3} [\sigma^{ij} \partial_i f \partial_j g + \sigma^{ij} v_i v_j f g] dv dx, \quad (1.23)$$

$$\mathcal{E}_\vartheta(f(t)) := \frac{1}{2} \|f(t)\|_{2,\vartheta}^2 + \int_0^t \|f(s)\|_{\sigma,\vartheta}^2 ds. \quad (1.24)$$

Remark 1.1. If $g \in C_v^1$, $f \in C_v^2$, and $\varphi \in C_v^2$ with a compact support in \mathbb{R}^3 , then

$$\begin{aligned} \int_{\mathbb{R}^3} (\bar{A}_g(f) + \bar{K}_g(f)) \varphi &= -\langle f, \varphi \rangle_\sigma - \int_{\mathbb{R}^3} \sigma_{\mu^{1/2}g} \partial_i f \partial_j \varphi - (a_g \cdot \nabla_v \varphi) f \\ &\quad + \left(K\varphi + \partial_i \sigma^i \varphi - \partial_i \left\{ \phi^{ij} * [\mu^{1/2} \partial_j g] \right\} \varphi + \left\{ \phi^{ij} * [v_i \mu^{1/2} \partial_j g] \right\} \varphi \right) f dv. \end{aligned}$$

We first consider the linearized Landau equation with a given g :

$$\partial_t f + v \cdot \nabla_v f + Lf = \Gamma(g, f). \quad (1.25)$$

If f satisfies (1.25), then for every $\vartheta \in \mathbb{R}$, $f^\vartheta := w^\vartheta f$ satisfies

$$\begin{aligned} \partial_t f^\vartheta + v \cdot \nabla_x f^\vartheta &= \bar{A}_g^\vartheta f^\vartheta + \bar{K}_g^\vartheta f, \\ f^\vartheta(0) &= w^\vartheta f_0 =: f_0^\vartheta, \end{aligned}$$

where

$$\bar{K}_g^\vartheta f = w^\vartheta \bar{K}_g f + \left(2 \frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}} \sigma_G^{ij} - \frac{\partial_{ij} w^\vartheta}{w^\vartheta} \sigma_G^{ij} - \frac{\partial_j w^\vartheta}{w^\vartheta} \partial_i \sigma_G^{ij} - \frac{\partial_i w^\vartheta}{w^\vartheta} a_g^i \right) f^\vartheta, \quad (1.26)$$

$$\bar{A}_g^\vartheta := \bar{A}_g - 2 \frac{\partial_i w^\vartheta}{w^\vartheta} \sigma_G^{ij} \partial_j. \quad (1.27)$$

Note that $\bar{A}_g^0 = \bar{A}_g$ and $f^0 = f$. Later we will derive an energy estimate for the equation:

$$h_t + v \cdot \nabla_x h = \bar{A}_g^\vartheta h \quad (1.28)$$

Here we introduce the main result.

Definition 1.2. Let $f(t, x, v) \in L^\infty((0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3, w^\vartheta(v) dt dx dv)$ be a periodic function in $x \in \mathbb{T}^3 = [-\pi, \pi]^3$ satisfying

$$\int_0^t \|f(s)\|_{\sigma, \vartheta}^2 ds < \infty. \quad (1.29)$$

We say that f is a weak solution of the Landau equation (1.5), (1.6) on $(0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3$ if for all $t \in (0, \infty)$ and all $\varphi \in C_{t,x,v}^{1,1,1}((0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3)$ such that $\varphi(t, x, v)$ is a periodic function in $x \in \mathbb{T}^3 = [-\pi, \pi]^3$ and $\varphi(t, x, \cdot)$ is compactly supported in \mathbb{R}^3 , it satisfies

$$\begin{aligned} &\iint_{\mathbb{T}^3 \times \mathbb{R}^3} f(t, x, v) \varphi(t, x, v) dx dv - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_0(x, v) \varphi(0, x, v) dx dv \\ &= -\langle f, \varphi \rangle_\sigma + \iiint_{(0,t) \times \mathbb{T}^3 \times \mathbb{R}^3} f(s, x, v) \left(\partial_s \varphi(s, x, v) + v \cdot \nabla_x \varphi(s, x, v) + a_f(s, x, v) \cdot \nabla_v \varphi(s, x, v) \right. \\ &\quad + K\varphi(s, x, v) + \partial_i \sigma^i(s, x, v) \varphi(s, x, v) - \partial_i \left\{ \phi^{ij} * [\mu^{1/2} \partial_j f] \right\} (s, x, v) \varphi(s, x, v) \\ &\quad \left. + \left\{ \phi^{ij} * [v_i \mu^{1/2} \partial_j f] \right\} (s, x, v) \varphi(s, x, v) \right) - \sigma_{\mu^{1/2}f}(s, x, v) \partial_i f(s, x, v) \partial_j \varphi(s, x, v) ds dx dv. \end{aligned} \quad (1.30)$$

Theorem 1.3 (Main results). *There exist ϑ' and $0 < \varepsilon_0 \ll 1$ such that for some $\vartheta \geq \vartheta'$ if f_0 satisfies*

$$\|f_0\|_{\infty, \vartheta} \leq \varepsilon_0, \quad \|f_{0t}\|_{\infty, \vartheta} + \|D_v f_0\|_{\infty, \vartheta} < \infty, \quad (1.31)$$

where $f_{0t} := -v \cdot \nabla_x f_0 + \bar{A}_{f_0} f_0$.

- (1) Then there exists a unique weak solution f of (1.5), (1.6) on $(0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3$.
- (2) Let $F(t, x, v) = \mu(v) + \sqrt{\mu}(v)f(t, x, v)$. If $F(0) \geq 0$, then $F(t) \geq 0$ for every $t \geq 0$.
- (3) Moreover for any $t > 0$, $\vartheta_0 \in \mathbb{N}$, and $\vartheta \geq \vartheta'$, there exist C , $C_{\vartheta, \vartheta_0}$, $l_0(\vartheta_0)$, and $0 < \alpha < 1$ such that f satisfies

$$\sup_{0 \leq s \leq \infty} \mathcal{E}_{\vartheta}(f(s)) \leq C 2^{2\vartheta} \mathcal{E}_{\vartheta}(0), \quad (1.32)$$

$$\|f(t)\|_{2, \vartheta} \leq C_{\vartheta, \vartheta_0} \mathcal{E}_{\vartheta+\vartheta_0/2}(0)^{1/2} \left(1 + \frac{t}{\vartheta_0}\right)^{-\vartheta_0/2}, \quad (1.33)$$

$$\|f(t)\|_{\infty, \vartheta} \leq C_{\vartheta, \vartheta_0} (1+t)^{-\vartheta_0} \|f_0\|_{\infty, \vartheta+l_0}, \quad (1.34)$$

$$\|f\|_{C^\alpha(\Omega)} \leq C (\|f_0\|_{\infty, \vartheta} + \|f_0\|_{\infty, \vartheta}), \quad (1.35)$$

and

$$\|D_v f\|_{L^\infty((0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq C (\|f_0\|_{\infty, \vartheta} + \|D_v f_0\|_{\infty, \vartheta} + \|f_0\|_{\infty, \vartheta}). \quad (1.36)$$

Motivated by the study of global well-posedness for the Landau equation in a bounded domain with physical boundary conditions, our current study is the first step to develop a $L^2 \rightarrow L^\infty$ framework with necessary analytic tools in a simpler periodic domain. There have been many results for Landau equations in either a periodic box or whole domain [2], [3], [6], [7], [8], [13], [15], [16], [17], [21], [22], [24], [25], and [26], in which high-order Sobolev norms can be employed. On the other hand, in a bounded domain, even with the velocity diffusion, the solutions can not be smooth up to the grazing set [18]. New mathematical tools involving weaker norms are needed to be developed. In the case for Boltzmann equations, a $L^2 \rightarrow L^\infty$ framework has been developed to construct unique global solutions in bounded domains [14].

Our work can be viewed as a similar $L^2 \rightarrow L^\infty$ approach for the Landau equation. Our techniques are inspired by recent remarkable progresses of [10], [11], and [19], in which a general machinery in the spirit of De Giorgi, has been developed to the Fokker-Planck equations, even to the Landau equation [19], to bootstrap L^∞ and Holder space $C^{0, \alpha}$ from a L^2 weak solution. Unfortunately, to our knowledge, there is still no construction for L^∞ global weak solutions to the Landau equation.

Our paper settles the global existence and uniqueness for a L^2 weak solution with a small weighted L^∞ perturbation of a Maxwellian initially. Our method is an intricate combination of different tools. Our starting point is a design of an iterating sequence

$$\begin{aligned} (\partial_t + v \cdot \nabla_x) f^{n+1} &= -L f^{n+1} + \Gamma(f^n, f^{n+1}) \\ &\equiv A_{f^n}(f^{n+1}) + K_{f^n}(f^{n+1}), \end{aligned}$$

where $A_{f^n}(f^{n+1})$ contains *all* the derivatives and $K_{f^n}(f^{n+1})$ has *no derivative* of f^n and f^{n+1} , so that f^n appears in the coefficients of the Landau operator for f^{n+1} . The crucial lemma states that if $\|f^n\|_\infty$ is sufficiently small, the main part of $A_{f^n}(f^{n+1})$ retains the same analytical properties of the linearized Landau operator A .

We first establish global energy estimates and time-decay under the assumption f^n is small, in Section 4.

Theorem 1.4. *Suppose that $\|g\|_\infty < \varepsilon$. Let $\vartheta \in 2^{-1}\mathbb{N} \cup \{0\}$ and f be a classical solution of (1.6), (1.7), and (1.25). Then there exist C and $\varepsilon = \varepsilon(\vartheta) > 0$ such that*

$$\sup_{0 \leq s < \infty} \mathcal{E}_{\vartheta}(f(s)) \leq C 2^{2\vartheta} \mathcal{E}_{\vartheta}(0), \quad (1.37)$$

and

$$\|f(t)\|_{2,\vartheta} \leq C_{\vartheta,k} (\mathcal{E}_{\vartheta+k/2}(0))^{1/2} \left(1 + \frac{t}{k}\right)^{-k/2} \quad (1.38)$$

for any $t > 0$ and $k \in \mathbb{N}$.

It is important to note that, thanks to the nonlinearity, the velocity weight can be arbitrarily strong. The proof of this step is a combination of energy estimates with positivity estimates for $\mathbf{P}f$ [9], [13] and a time-decay estimate [24], but in the absence of high-order Sobolev regularity.

We next bootstrap such a L^2 bound to a L^∞ bound.

Theorem 1.5. *Suppose that $\|g\|_\infty < \varepsilon$. Let f be a weak solution of (1.6), (1.7), and (1.25) in a periodic box and $\vartheta \in \mathbb{N} \cup \{0\}$, $\vartheta_0 \in \mathbb{N}$. Then there exist ε , $l_0(\vartheta_0) > 0$ and $C_{\vartheta,\vartheta_0}$ such that*

$$\|f(t)\|_{\infty,\vartheta+\vartheta_0} \leq C_{\vartheta,\vartheta_0} (1+t)^{-\vartheta_0} \|f_0\|_{\infty,\vartheta+l_0}. \quad (1.39)$$

It is important to note that even though there is a finite loss of velocity weight, we are still able to close the estimates thanks to the strong gain of velocity weight in (1.38). The proof of such a L^∞ estimate locally in x and v is an adaptation of recent work of [11], [19]. It is well-known that the Landau operator is delicate to study and estimate for large velocities. Together with the maximum principle of the Landau operator as well as strong time decay for L^2 norm in (1.38), we are able to control the ‘tails’ of solutions for large velocities, and obtain global (in x and v) L^∞ estimate.

Unfortunately, unlike in the Boltzmann case (see [14]), in order to establish the convergence of $\{f^n\}$ and more importantly, uniqueness of our solution, such a L^∞ bound is not sufficient due to the presence of velocity derivative in the nonlinear Landau equation. We need to further control $\|\nabla_v f^n\|_\infty$ as in Lemma 8.2, which follows from S_p estimates established in [23]. One crucial requirement for such S_p estimates (as the classical $W^{2,p}$ estimate in the elliptic theory), is the $C^{0,\alpha}$ estimate (uniform in x and v) for the coefficients containing f^n . We establish

Theorem 1.6. *Let f be a solution of (1.6), (1.7), and (1.25). Then there exist ε , ϑ' , C , $\alpha > 0$ such that if g satisfies (2.4), then we have*

$$\|f\|_{C^\alpha(\Omega)} \leq C(f_0),$$

where

$$C(f_0) = C(\|f_{0t}\|_{\infty,\vartheta} + \|f_0\|_{\infty,\vartheta}). \quad (1.40)$$

Again, we follow the methods in [10], [11], and [19] to establish such an estimate locally in (x, v) , and use a delicate change of coordinates (6.16) locally to capture precisely the isotropic behavior of the Landau operator, thanks to Lemma 2.4 and our strong weighted L^∞ estimates to obtain uniform $C^{0,\alpha}$ estimate. It is well-known that the Landau equation is degenerate for $|v| \rightarrow \infty$ and our strong energy estimate provides the control of velocity (tails) of the Landau solutions. An additional regularity condition $\|f_{0t}\|_{\infty,\theta} < +\infty$ is needed for such a Holder estimate, but no smallness is required. A further bound $\|f_{0v}\|_{\infty,\theta} < +\infty$ is needed to apply the S_p theory in a non-divergent form.

Such a $L^2 \rightarrow L^\infty$ framework is robust and is currently being applied to the study of several other problems in the kinetic theory.

2. BASIC ESTIMATES

For the reader's convenience, we summarize and modify some basic estimates. We will adapt techniques in [13].

For every $v, \nu \in \mathbb{R}^3$, define

$$D_u(\nu; v) := \nu^T \sigma_u(v) \nu \quad (2.1)$$

and P_v is the projection onto the vector v as

$$P_v g := \sum \langle g_j, v_j \rangle \frac{v_i}{|v|^2}, \quad 1 \leq i \leq 3. \quad (2.2)$$

Proposition 2.1. *There exists a uniform constant C such that for every function f and a constant $\vartheta \in \mathbb{R}$, we have*

$$\|f\|_{2, \vartheta} \leq C \|f\|_{\infty, \vartheta+2}.$$

Proof.

$$\begin{aligned} \|f\|_{2, \vartheta}^2 &= \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (w^\vartheta(v) f(x, v))^2 dx dv \\ &= \iint_{\mathbb{T}^3 \times \mathbb{R}^3} w^{-4}(v) (w^{\vartheta+2}(v) f(x, v))^2 dx dv \\ &\leq \|f\|_{\infty, \vartheta+2}^2 \iint_{\mathbb{T}^3 \times \mathbb{R}^3} w^{-4}(v) dx dv \\ &\leq C \|f\|_{\infty, \vartheta+2}^2, \end{aligned}$$

for some constant $C > 0$. □

Lemma 2.2 (Lemma 2 in [13]). *Let $\vartheta > -3$, $a(v) \in C^\infty(\mathbb{R}^3)$ and $b(v) \in C^\infty(\mathbb{R}^3 \setminus \{0\})$. Assume for any positive multi-index β , there is $C_\beta > 0$ such that*

$$\begin{aligned} |\partial_\beta a(v)| &\leq C_\beta |v|^{\vartheta-|\beta|}, \\ |\partial_\beta b(v)| &\leq C_\beta e^{-\tau_\beta |v|^2}, \end{aligned}$$

with some $\tau_\beta > 0$. Then there is $C_\beta^ > 0$ such that*

$$|\partial_\beta [a * b](v)| \leq C_\beta^* [1 + |v|]^{\vartheta-\beta}.$$

Lemma 2.3 (Lemma 3 in [13]). *If $u = \mu$ or $\sqrt{\mu}$, then*

$$D_u(\nu; v) = \lambda_1(v) |P_v \nu|^2 + \lambda_2(v) |(I - P_v) \nu|^2. \quad (2.3)$$

Moreover, there exists C such that

$$\frac{1}{C} (1 + |v|)^{-3} \leq \lambda_1(v) \leq C (1 + |v|)^{-3},$$

and

$$\frac{1}{C} (1 + |v|)^{-1} \leq \lambda_2(v) \leq C (1 + |v|)^{-1}.$$

We can derive upper and lower bounds of eigenvalues for $\sigma + \sigma_{\sqrt{\mu}g}$ by adapting ideas in the proof of Theorem 3 in [13].

Lemma 2.4. *Let g be a given function in $L^\infty((0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3)$ and $G = \mu + \sqrt{\mu}g$. Let σ_G be the matrix defined as in (1.12). Then there exists $0 < \varepsilon \ll 1$ such that if g satisfies*

$$\sup_{0 \leq s \leq \infty} \|g(s)\|_\infty \leq \varepsilon \quad (2.4)$$

then

$$\begin{aligned} D_G(\nu; v) &\geq \frac{1}{2C} \left((1 + |v|)^{-3} |P_v \nu|^2 + (1 + |v|)^{-1} |(I - P_v) \nu|^2 \right), \\ D_G(\nu; v) &\leq 2C \left((1 + |v|)^{-3} |P_v \nu|^2 + (1 + |v|)^{-1} |(I - P_v) \nu|^2 \right), \end{aligned}$$

for every $v \in \mathbb{R}^3$. Thus $\sigma_G(v)$ has three non-negative eigenvalues. Moreover, $\lambda(v)$, eigenvalue of $\sigma_G(v)$, has the following estimate

$$\frac{1}{C} (1 + |v|)^{-3} \leq \lambda(v) \leq C (1 + |v|)^{-1},$$

for some constant $C > 0$.

Proof. Let $u = \sqrt{\mu}g$. Then we claim that there exists $C' > 0$ such that

$$|D_u(\nu; v)| \leq C' \|g\|_\infty \left((1 + |v|)^{-3} |P_v \nu|^2 + (1 + |v|)^{-1} |(I - P_v) \nu|^2 \right). \quad (2.5)$$

Consider

$$\begin{aligned} D_u(\nu; v) &= \sum_{i,j} \int_{2|v'| > |v|} \nu_i \nu_j \phi^{ij}(v - v') \sqrt{\mu}(v') g(v') dv' + \sum_{i,j} \int_{2|v'| \leq |v|} \nu_i \nu_j \phi^{ij}(v - v') \sqrt{\mu}(v') g(v') dv' \\ &= (I) + (II). \end{aligned}$$

Note that for $2|v'| > |v|$, $\sqrt{\mu}(v') \leq C' \mu(v'/4) \mu(v/4)$. Therefore,

$$\begin{aligned} |(I)| &\leq C' \mu\left(\frac{v}{4}\right) \|g\|_\infty |\nu|^2 \int_{\mathbb{R}^3} \phi^{ij}(v - v') \mu\left(\frac{v'}{4}\right) dv' \\ &\leq C'^{-1} \mu\left(\frac{v}{4}\right) \|g\|_\infty |\nu|^2. \end{aligned} \quad (2.6)$$

To control (II), we expand $\phi^{ij}(v - v')$ to get

$$\phi^{ij}(v - v') = \sum_k \partial_k \phi^{ij}(v) v'_k + \frac{1}{2} \sum_{k,l} \partial_{kl} \phi^{ij}(\bar{v}) v'_k v'_l,$$

for some \bar{v} in a line segment of v and $v - v'$. Then we have

$$\begin{aligned} (II) &= \sum_{i,j} \nu_i \nu_j \phi^{ij}(v) \int_{2|v'| \leq |v|} \sqrt{\mu}(v') g(v') dv' \\ &\quad - \sum_{i,j} \nu_i \nu_j \sum_k \partial_k \phi^{ij}(v) \int_{2|v'| \leq |v|} v'_k \sqrt{\mu}(v') g(v') dv' \\ &\quad + \frac{1}{2} \sum_{i,j} \int_{2|v'| \leq |v|} \nu_i \nu_j \sum_{k,l} \partial_{kl} \phi^{ij}(\bar{v}) v'_k v'_l \sqrt{\mu}(v') g(v') dv' \\ &= (II)_1 + (II)_2 + (II)_3. \end{aligned}$$

Since

$$\sum_i \phi^{ij}(v)v_i = \sum_j \phi^{ij}(v)v_j = 0,$$

we have

$$\begin{aligned} |(II)_1| &= ((I - P_v)\nu)^T \phi(v) ((I - P_v)\nu) \int_{2|v'| \leq |v|} \sqrt{\mu}(v') g(v') dv' \\ &\leq C \|g\|_\infty (1 + |v|)^{-1} |(I - P_v)\nu|^2. \end{aligned} \quad (2.7)$$

Note that

$$\sum_{i,j} \partial_k \phi^{ij}(v) v_i v_j = 0.$$

Therefore

$$\begin{aligned} |(II)_2| &\leq \left| \sum_k ((I - P_v)\nu)^T \partial_k \phi(v) ((I - P_v)\nu) \int_{2|v'| \leq |v|} v'_k \sqrt{\mu}(v') g(v') dv' \right| \\ &\quad + 2 \left| \sum_k (P_v \nu)^T \partial_k \phi(v) ((I - P_v)\nu) \int_{2|v'| \leq |v|} v'_k \sqrt{\mu}(v') g(v') dv' \right| \\ &\leq C \|g\|_\infty (1 + |v|)^{-2} (|(I - P_v)\nu|^2 + |P_v \nu| |(I - P_v)\nu|) \\ &\leq C \|g\|_\infty ((1 + |v|)^{-3} |P_v \nu|^2 + (1 + |v|)^{-1} |(I - P_v)\nu|^2). \end{aligned} \quad (2.8)$$

Since \bar{v} is in $(v, v - v')$ and $2|v'| \leq |v|$, we have $|v|/2 \leq |\bar{v}| \leq 3|v|/2$. Therefore, $\partial_{kl} \phi^{ij}(\bar{v}) \leq C' |v|^{-3}$. Thus we have

$$|(II)_3| \leq C' \|g\|_\infty (1 + |v|)^{-3} |\nu|^2. \quad (2.9)$$

Combining (2.6) - (2.9), we have (2.5).

Now we can compute $D_G(\nu; v)$. Since $\varepsilon > 0$ is a given small enough constant, from (2.3), (2.5), we have

$$\begin{aligned} D_G(\nu; v) &\geq \frac{1}{2C} ((1 + |v|)^{-3} |P_v \nu|^2 + (1 + |v|)^{-1} |(I - P_v)\nu|^2), \\ D_G(\nu; v) &\leq 2C ((1 + |v|)^{-3} |P_v \nu|^2 + (1 + |v|)^{-1} |(I - P_v)\nu|^2). \end{aligned}$$

Therefore,

$$\frac{1}{2C} (1 + |v|)^{-3} \leq \lambda \leq 2C (1 + |v|)^{-1}.$$

□

Lemma 2.5 (Corollary 1 in [13]). *There exists $c = c_\vartheta > 0$, such that*

$$\begin{aligned} |g|_{\sigma, \vartheta}^2 &\geq c \left\{ \left| w^\vartheta [1 + |v|]^{-3/2} \{P_v \partial_i g\} \right|_2^2 + \left| w^\vartheta [1 + |v|]^{-1/2} \{[I - P_v] \partial_i g\} \right|_2^2 \right. \\ &\quad \left. + \left| w^\vartheta [1 + |v|]^{-1/2} g \right|_2^2 \right\}. \end{aligned}$$

Lemma 2.6 (Lemma 5 in [13]). *Let L , K , and σ^i be defined as in (1.8), (1.13), and (1.10). Let $\vartheta \in \mathbb{R}$. For any $m > 1$, there is $0 < C(m) < \infty$, such that*

$$\begin{aligned} & |\langle w^{2\vartheta} \partial_i \sigma^i g_1, g_2 \rangle| + |\langle w^{2\vartheta} K g_1, g_2 \rangle| \\ & \leq \frac{C}{m} |g_1|_{\sigma, \vartheta} |g_2|_{\sigma, \vartheta} + C(m) \left\{ \int_{|v| \leq C(m)} |w^\vartheta g_1|^2 dv \right\}^{1/2} \left\{ \int_{|v| \leq C(m)} |w^\vartheta g_2|^2 dv \right\}^{1/2}. \end{aligned}$$

Moreover, there is $\delta > 0$, such that

$$\langle Lg, g \rangle \geq \delta |(I - \mathbf{P})g|_\sigma^2.$$

Lemma 2.7 (Lemma 6 in [13]). *Let L , A , and K be defined as in (1.8), (1.9), and (1.10). Let $\vartheta \in \mathbb{R}$ and $|\beta| \geq 0$. For small $\delta > 0$, there exists $C_\delta = C_\delta(\vartheta) > 0$ such that*

$$-\langle w^{2\vartheta} Ag, g \rangle \geq |g|_{\sigma, \vartheta}^2 - \delta |g|_{\sigma, \vartheta}^2 - C_\delta |\mu g|_2^2,$$

$$|\langle w^{2\vartheta} K g_1, g_2 \rangle| \leq \{\delta |g_1|_{\sigma, \vartheta} + C_\delta |\mu g_1|_2\} |g_2|_{\sigma, \vartheta}.$$

Thus we have

$$\frac{1}{2} |g|_{\sigma, \vartheta}^2 - C_\vartheta |g|_\sigma^2 \leq \langle w^{2\vartheta} Lg, g \rangle \leq \frac{3}{2} |g|_{\sigma, \vartheta}^2 + C_\vartheta |g|_\sigma^2.$$

For the nonlinear estimate in Theorem 3 [13], they estimated

$$\left(w^{2\vartheta} \Gamma[g_1, g_2], g_3 \right)$$

in terms of $\|g_i\|_{2, \vartheta}$ and $\|g_i\|_{\sigma, \vartheta}$ for $i = 1, 2, 3$ and $\vartheta \geq 0$. To get such a L^2 estimate, they need a higher-order regularity like $\|D_\beta^\alpha g_i\|_{2, \vartheta}$ and $\|D_\beta^\alpha g_i\|_{\sigma, \vartheta}$ for $i = 1, 2, 3$, $\vartheta \geq 0$, $|\alpha| + |\beta| \leq N$ and $N \geq 8$.

The following lemma is a refinement of Theorem 3 in [13]. First, the range of ϑ is extended to \mathbb{R} . Second, we estimate the nonlinear term in terms of $\|\cdot\|_\infty$, $\|\cdot\|_{2, \vartheta}$, and $\|\cdot\|_{\sigma, \vartheta}$ without a higher-order regularity.

Theorem 2.8. *Let Γ be defined as in (1.11).*

(1) *For every $\vartheta \in \mathbb{R}$, there exists C_ϑ such that*

$$|\langle w^{2\vartheta} \Gamma[g_1, g_2], g_3 \rangle| \leq C_\vartheta |g_1|_\infty |g_2|_{\sigma, \vartheta} |g_3|_{\sigma, \vartheta}, \quad (2.10)$$

and

$$\left| \left(w^{2\vartheta} \Gamma[g_1, g_2], g_3 \right) \right| \leq C_\vartheta \|g_1\|_\infty \|g_2\|_{\sigma, \vartheta} \|g_3\|_{\sigma, \vartheta}. \quad (2.11)$$

(2) *There exists $\bar{\vartheta} < 0$ such that for any $\vartheta \leq \bar{\vartheta}$,*

$$\left| \left(w^{2\vartheta} \Gamma[g_1, g_2], g_3 \right) \right| \leq C_\vartheta \min\{\|g_1\|_{2, \vartheta}, \|g_1\|_{\sigma, \vartheta}\} (\|g_2\|_\infty + \|D_v g_2\|_\infty) \|g_3\|_{\sigma, \vartheta}. \quad (2.12)$$

Proof. (1) By the integration by parts, we have

$$\begin{aligned}
|\langle w^{2\vartheta} \Gamma[g_1, g_2], g_3 \rangle| &\leq |\langle \partial_i w^{2\vartheta} \{\phi^{ij} * [\sqrt{\mu} g_1]\} \partial_j g_2, g_3 \rangle| \\
&\quad + |\langle w^{2\vartheta} \{\partial_j \phi^{ij} * [\sqrt{\mu} g_1]\} g_2, \partial_i g_3 \rangle| \\
&\quad + |\langle \partial_i w^{2\vartheta} \{\partial_j \phi^{ij} * [\sqrt{\mu} g_1]\} g_2, g_3 \rangle| \\
&\quad + |\langle \partial_i w^{2\vartheta} \{\phi^{ij} * [v_j \sqrt{\mu} g_1]\} g_2, g_3 \rangle| \\
&\quad + |\langle w^{2\vartheta} \{\partial_j \phi^{ij} * [v_i \sqrt{\mu} g_1]\} g_2, g_3 \rangle| \\
&\quad + |\langle w^{2\vartheta} \{\phi^{ij} * [\partial_j \{v_i \sqrt{\mu}\} g_1]\} g_2, g_3 \rangle| \\
&\quad + |\langle w^{2\vartheta} \{\phi^{ij} * [v_i \sqrt{\mu} g_1]\} \partial_j g_2, g_3 \rangle| \\
&\quad + |\langle w^{2\vartheta} \{\phi^{ij} * [v_j \sqrt{\mu} g_1]\} g_2, \partial_i g_3 \rangle| \\
&\quad + |\langle w^{2\vartheta} \{\phi^{ij} * [\sqrt{\mu} g_1]\} \partial_j g_2, \partial_i g_3 \rangle| \\
&= (I) + (II) + \cdots + (IX),
\end{aligned} \tag{2.13}$$

where ϕ is the matrix defined as in (1.2). Clearly, $|\partial_i w^{2\vartheta}| \leq C_\vartheta (1 + |v|)^{-1} w^{2\vartheta}$ and by Lemma 2.2, we have

$$\begin{aligned}
|\phi^{ij} * [\sqrt{\mu} g_1]| + |\phi^{ij} * [v_i \sqrt{\mu} g_1]| + |\phi^{ij} * [v_j \sqrt{\mu} g_1]| + |\phi^{ij} * [\partial_j \{v_i \sqrt{\mu}\} g_1]| &\leq C(1 + |v|)^{-1} \|g_1\|_\infty, \\
|\partial_j \phi^{ij} * [\sqrt{\mu} g_1]| + |\partial_j \phi^{ij} * [v_i \sqrt{\mu} g_1]| &\leq C(1 + |v|)^{-2} \|g_1\|_\infty.
\end{aligned}$$

Therefore, by Lemma 2.5 and the Hölder inequality,

$$\begin{aligned}
(I) &\leq C_\vartheta |g_1|_\infty |\langle w^\vartheta (1 + |v|)^{-3/2} \partial_j g_2, w^\vartheta (1 + |v|)^{-1/2} g_3 \rangle| \\
&\leq C_\vartheta |g_1|_\infty |g_2|_{\sigma, \vartheta} |g_3|_{\sigma, \vartheta}, \\
(II) &\leq C |g_1|_\infty |\langle w^\vartheta (1 + |v|)^{-1/2} g_2, w^\vartheta (1 + |v|)^{-3/2} \partial_i g_3 \rangle| \\
&\leq C |g_1|_\infty |g_2|_{\sigma, \vartheta} |g_3|_{\sigma, \vartheta}, \\
(III) + (IV) + (V) + (VI) &\leq C_\vartheta |g_1|_\infty |\langle w^\vartheta (1 + |v|)^{-1/2} g_2, w^\vartheta (1 + |v|)^{-1/2} g_3 \rangle| \\
&\leq C_\vartheta |g_1|_\infty |g_2|_{\sigma, \vartheta} |g_3|_{\sigma, \vartheta}.
\end{aligned}$$

By (2.5) and the Hölder inequality,

$$\begin{aligned}
(VII) &\leq C |g_1|_\infty \int w^{2\vartheta} \left| (1 + |v|)^{-3/2} |P_v \partial_j g_2| + (1 + |v|)^{-1/2} |(I - P_v) \partial_j g_2| \right| (1 + |v|)^{-1/2} |g_3| dv \\
&\leq C |g_1|_\infty |g_2|_{\sigma, \vartheta} |g_3|_{\sigma, \vartheta}, \\
(VIII) &\leq C |g_1|_\infty \int w^{2\vartheta} (1 + |v|)^{-1/2} |g_2| \left| (1 + |v|)^{-3/2} |P_v \partial_j g_3| + (1 + |v|)^{-1/2} |(I - P_v) \partial_j g_3| \right| dv \\
&\leq C |g_1|_\infty |g_2|_{\sigma, \vartheta} |g_3|_{\sigma, \vartheta},
\end{aligned}$$

and

$$\begin{aligned}
(IX) &\leq C |g_1|_\infty \int w^{2\vartheta} \left| (1 + |v|)^{-3/2} |P_v \partial_j g_2| + (1 + |v|)^{-1/2} |(I - P_v) \partial_j g_2| \right| \\
&\quad \times \left| (1 + |v|)^{-3/2} |P_v \partial_j g_3| + (1 + |v|)^{-1/2} |(I - P_v) \partial_j g_3| \right| dv \\
&\leq C |g_1|_\infty |g_2|_{\sigma, \vartheta} |g_3|_{\sigma, \vartheta}.
\end{aligned}$$

Thus we obtain (2.10). By applying the Hölder inequality to (2.10),

$$\begin{aligned} \left| \left(w^{2\vartheta} \Gamma[g_1, g_2], g_3 \right) \right| &= \int |\langle w^{2\vartheta} \Gamma[g_1, g_2], g_3 \rangle| dx \\ &\leq \int C_\vartheta |g_1|_\infty |g_2|_{\sigma, \vartheta} |g_3|_{\sigma, \vartheta} dx \\ &\leq C_\vartheta \|g_1\|_\infty \|g_2\|_{\sigma, \vartheta} \|g_3\|_{\sigma, \vartheta}. \end{aligned}$$

Thus we have (2.11).

(2) By the integration by parts again, we have

$$\begin{aligned} |\langle w^{2\vartheta} \Gamma[g_1, g_2], g_3 \rangle| &:= |\langle w^{2\vartheta} \{\phi^{ij} * [\mu^{1/2} g_1]\} \partial_j g_2, \partial_i g_3 \rangle| \\ &\quad + |\langle w^{2\vartheta} \{\phi^{ij} * [v_i \mu^{1/2} g_1]\} \partial_j g_2, g_3 \rangle| \\ &\quad + |\langle w^{2\vartheta} \{\phi^{ij} * [\mu^{1/2} \partial_j g_1]\} g_2, \partial_i g_3 \rangle| \\ &\quad + |\langle w^{2\vartheta} \{\phi^{ij} * [v_i \mu^{1/2} \partial_j g_1]\} g_2, g_3 \rangle| \\ &\quad + |\langle \partial_i w^{2\vartheta} \{\phi^{ij} * [\mu^{1/2} g_1]\} \partial_j g_2, g_3 \rangle| \\ &\quad + |\langle \partial_i w^{2\vartheta} \{\phi^{ij} * [\mu^{1/2} \partial_j g_1]\} g_2, g_3 \rangle| \\ &= (i) + (ii) + \cdots + (vi). \end{aligned} \tag{2.14}$$

By the Hölder inequality and the integration by parts, we have

$$\begin{aligned} &|\langle \{\phi^{ij} * [\mu^{1/2} g_1]\} \rangle| + |\langle \{\phi^{ij} * [v_i \mu^{1/2} g_1]\} \rangle| + |\langle \{\phi^{ij} * [\mu^{1/2} \partial_j g_1]\} \rangle| + |\langle \{\phi^{ij} * [v_i \mu^{1/2} \partial_j g_1]\} \rangle| \\ &\leq C_\vartheta (1 + |v|)^{-1} \min\{|g_1|_{2, \vartheta}, |g_1|_{\sigma, \vartheta}\}. \end{aligned}$$

Let $\bar{\vartheta} := -2$, then by applying the Hölder inequality to (i) and Lemma 2.5, we have

$$\begin{aligned} (i) &\leq C_\vartheta \min\{|g_1|_{2, \vartheta}, |g_1|_{\sigma, \vartheta}\} \left| \int_{\mathbb{R}^3} w^{2\vartheta} (1 + |v|)^{-1} \partial_j g_2(v) \partial_i g_3(v) dv \right| \\ &\leq C_\vartheta \min\{|g_1|_{2, \vartheta}, |g_1|_{\sigma, \vartheta}\} |\partial_j g_2|_\infty \left(\int_{\mathbb{R}^3} (1 + |v|)^{2\vartheta+1} dv \right)^{1/2} \left(\int_{\mathbb{R}^3} w^{2\vartheta} (1 + |v|)^{-3} |\partial_i g_3|^2 dv \right)^{1/2} \\ &\leq C_\vartheta \min\{|g_1|_{2, \vartheta}, |g_1|_{\sigma, \vartheta}\} |D_v g_2|_\infty |g_3|_{\sigma, \vartheta}. \end{aligned}$$

Similarly,

$$(ii) + (iii) + \cdots + (vi) \leq C_\vartheta \min\{|g_1|_{2, \vartheta}, |g_1|_{\sigma, \vartheta}\} (|g_2|_\infty + |D_v g_2|_\infty) |g_3|_{\sigma, \vartheta}.$$

Therefore, by the Hölder inequality again, we have

$$\begin{aligned} \left| \left(w^{2\vartheta} \Gamma[g_1, g_2], g_3 \right) \right| &= \int |\langle w^{2\vartheta} \Gamma[g_1, g_2], g_3 \rangle| dx \\ &\leq \int C_\vartheta \min\{|g_1|_{2, \vartheta}, |g_1|_{\sigma, \vartheta}\} (|g_2|_\infty + |D_v g_2|_\infty) |g_3|_{\sigma, \vartheta} dx \\ &\leq C_\vartheta \min\{\|g_1\|_{2, \vartheta}, \|g_1\|_{\sigma, \vartheta}\} (\|g_2\|_\infty + \|D_v g_2\|_\infty) \|g_3\|_{\sigma, \vartheta}. \end{aligned}$$

□

Lemma 2.9. *Let \bar{K}_g^ϑ be defined as in (1.26). Suppose that g satisfies the assumption in Lemma 2.4. Then there exists $C = C_\vartheta > 0$ such that for every $N, M > 0$,*

$$\|\bar{K}_g^\vartheta f\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C \|f^\vartheta\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}, \quad (2.15)$$

$$\|\bar{K}_g^\vartheta 1_{|v|>M} f\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C(1+M)^{-1} \|f^\vartheta\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}, \quad (2.16)$$

and

$$\|\bar{K}_g^\vartheta f\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} \leq CN^2 \|f^\vartheta\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} + \frac{C}{N} \|f^\vartheta\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}. \quad (2.17)$$

Proof. Since

$$\begin{aligned} \bar{K}_g^\vartheta f &= w^\vartheta \bar{K}_g f + \left(2 \frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}} \sigma_G^{ij} - \frac{\partial_{ij} w^\vartheta}{w^\vartheta} \sigma_G^{ij} - \frac{\partial_j w^\vartheta}{w^\vartheta} \partial_i \sigma_G^{ij} - \frac{\partial_i w^\vartheta}{w^\vartheta} a_g^i \right) f^\vartheta \\ &= w^\vartheta K f + \partial_i \sigma^i f^\vartheta - v \cdot \sigma v f^\vartheta - \partial_i \left\{ \phi^{ij} * [\mu^{1/2} \partial_j g] \right\} f^\vartheta + \left\{ \phi^{ij} * [v_i \mu^{1/2} \partial_j g] \right\} f^\vartheta \\ &\quad + \left(2 \frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}} \sigma_G^{ij} - \frac{\partial_{ij} w^\vartheta}{w^\vartheta} \sigma_G^{ij} - \frac{\partial_j w^\vartheta}{w^\vartheta} \partial_i \sigma_G^{ij} - \frac{\partial_i w^\vartheta}{w^\vartheta} a_g^i \right) f^\vartheta, \end{aligned}$$

where ϕ , K , σ_G , σ^i are defined as in (1.2), (1.10), (1.12), and (1.13) with $G = \mu + \mu^{1/2}g$, by Lemma 2.2 and 2.3, we have

$$\begin{aligned} |\partial_i \sigma^i(v)| + |v \cdot \sigma v| + \left| \partial_i \left\{ \phi^{ij} * [\mu^{1/2} \partial_j g](v) \right\} \right| + \left| \left\{ \phi^{ij} * [v_i \mu^{1/2} \partial_j g](v) \right\} \right| &\leq C(1+|v|)^{-1}, \\ \left| 2 \frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}} \sigma_G^{ij} \right| + \left| \frac{\partial_{ij} w^\vartheta}{w^\vartheta} \sigma_G^{ij} \right| + \left| \frac{\partial_j w^\vartheta}{w^\vartheta} \partial_i \sigma_G^{ij} \right| + \left| \frac{\partial_i w^\vartheta}{w^\vartheta} a_g^i \right| &\leq C(1+|v|)^{-1}. \end{aligned}$$

Thus it is sufficient to show that $w^\vartheta K f$ also satisfies (2.15) - (2.17).

After the integration by parts, we have

$$\begin{aligned} w^\vartheta K f &= -w^\vartheta \mu^{-1/2} \partial_i \left\{ \mu \left[\phi^{ij} * \left\{ \mu^{1/2} [\partial_j f + v_j f] \right\} \right] \right\} \\ &= 2w^\vartheta v_i \mu \left[\phi^{ij} * \left\{ \mu^{1/2} [\partial_j f + v_j f] \right\} \right] - w^\vartheta \mu^{1/2} \left[\partial_i \phi^{ij} * \left\{ \mu^{1/2} [\partial_j f + v_j f] \right\} \right] \\ &= 2w^\vartheta v_i \mu \left[\phi^{ij} * (v_j \mu^{1/2} f) \right] - 2w^\vartheta v_i \mu \left[\phi^{ij} * (\partial_j \mu^{1/2} f) \right] + 2w^\vartheta v_i \mu \left[\partial_j \phi^{ij} * (\mu^{1/2} f) \right] \\ &\quad - w^\vartheta \mu^{1/2} \left[\partial_i \phi^{ij} * (v_j \mu^{1/2} f) \right] + w^\vartheta \mu^{1/2} \left[\partial_i \phi^{ij} * (\partial_j \mu^{1/2} f) \right] - w^\vartheta \mu^{1/2} \left[\partial_{ij} \phi^{ij} * (\mu^{1/2} f) \right] \\ &= 4w^\vartheta v_i \mu \left[\phi^{ij} * (v_j \mu^{1/2} f) \right] + 2w^\vartheta v_i \mu \left[\partial_j \phi^{ij} * (\mu^{1/2} f) \right] \\ &\quad - w^\vartheta 2\mu^{1/2} \left[\partial_i \phi^{ij} * (v_j \mu^{1/2} f) \right] - w^\vartheta \mu^{1/2} \left[\partial_{ij} \phi^{ij} * (\mu^{1/2} f) \right] \\ &= 4w^\vartheta v_i \mu \left[\phi^{ij} * (v_j w^{-\vartheta} \mu^{1/2} f^\vartheta) \right] + 2w^\vartheta v_i \mu \left[\partial_j \phi^{ij} * (w^{-\vartheta} \mu^{1/2} f^\vartheta) \right] \\ &\quad - w^\vartheta 2\mu^{1/2} \left[\partial_i \phi^{ij} * (v_j w^{-\vartheta} \mu^{1/2} f^\vartheta) \right] - w^\vartheta \mu^{1/2} \left[\partial_{ij} \phi^{ij} * (w^{-\vartheta} \mu^{1/2} f^\vartheta) \right] \\ &= (I) + (II) + (III) + (VI). \end{aligned}$$

Applying Lemma 2.2 to $(I) + (II) + (III)$, we have $(I) + (II) + (III) \leq C \|f^\vartheta\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}$. Note that $\partial_{ij} \phi^{ij} * (\mu^{1/2} f) = -8\pi \mu^{1/2} f$. Thus we also have $(VI) \leq C \|f^\vartheta\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}$. Therefore, we have (2.15).

Since every convolution term of Kf contains $\mu^{1/2}$, we have (2.16).

For (2.17), clearly we have

$$\|1_{|v|\geq N} w^\vartheta Kf\|_2 \leq \frac{C}{N} \|w^\vartheta f\|_\infty. \quad (2.18)$$

Now we will estimate $\|1_{|v|<N} w^\vartheta Kf\|_2$. First, consider $\|1_{|v|<N} v_i w^\vartheta \mu [\partial_j \phi^{ij} * w^{-\vartheta} \mu^{1/2} f^\vartheta]\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)}$.

$$\begin{aligned} & \left\| 1_{|v|<N} v_i w^\vartheta \mu \left[\partial_j \phi^{ij} * w^{-\vartheta} \mu^{1/2} f^\vartheta \right] \right\|_2^2 \\ &= \int_{\mathbb{T}^3} \int_{|v|<N} \left(\int \partial_j \phi^{ij}(v-v') v_i w^\vartheta(v) \mu(v) \mu^{1/2}(v') w^{-\vartheta}(v') f^\vartheta(v') dv' \right)^2 dv dx \\ &\leq C \left(\int_{\mathbb{T}^3} \int_{|v|<N} \left(\int_{1/N < |v-v'| < 2N} dv' \right)^2 dv dx \right. \\ &\quad \left. + \int_{\mathbb{T}^3} \int_{|v|<N} \left(\int_{|v-v'|>2N} dv' \right)^2 dv dx \right. \\ &\quad \left. + \int_{\mathbb{T}^3} \int_{|v|<N} \left(\int_{|v-v'|<1/N} dv' \right)^2 dv dx \right) \\ &= (i) + (ii) + (iii). \end{aligned}$$

Since $|\partial_j \phi^{ij}(v-v')| \leq C|v-v'|^{-2}$, by the Minkowski and Hölder inequality,

$$\begin{aligned} (i) &\leq \int_{\mathbb{T}^3} \left(\int_{|v'|<3N} \left(\int_{1/N < |v-v'| < 2N} |v-v'|^{-4} w^{2\vartheta}(v) v_i^2 \mu^2(v) w^{-2\vartheta}(v') \mu(v') (f^\vartheta)^2(v') dv \right)^{1/2} dv' \right)^2 dx \\ &= \int_{\mathbb{T}^3} \left(\int_{|v'|<3N} w^{-\vartheta}(v') \mu^{1/2}(v') f^\vartheta(v') \left(\int_{1/N < |v-v'| < 2N} |v-v'|^{-4} w^{2\vartheta}(v) v_i^2 \mu^2(v) dv \right)^{1/2} dv' \right)^2 dx \\ &\leq C \int_{\mathbb{T}^3} \left(\int_{|v'|<3N} \mu^{1/2}(v') (f^\vartheta)^2(v') dv' \right) \left(\int_{|v'|<3N} \int_{1/N < |v-v'| < 2N} |v-v'|^{-4} w^{2\vartheta}(v) v_i^2 \mu^2(v) dv dv' \right) dx \\ &\leq CN^4 \|f^\vartheta\|_2^2. \end{aligned}$$

Note that if $|v| < N$ and $|v-v'| > 2N$, then $|v'| > N$. Since the integrand of (ii) contains a Maxwellian and $|v'| > N$, for every $\beta > 0$ we have

$$(ii) \leq \frac{C_\beta}{N^{2\beta}} \|f^\vartheta\|_\infty^2.$$

Finally,

$$(iii) \leq C \|f^\vartheta\|_\infty^2 \iint_{\mathbb{T}^3 \times [0, \infty)} \int_{\mathbb{R}^3} \mu(v) \left(\int_{|v-v'|<1/N} |v-v'|^{-2} dv' \right)^2 dv dx dt \leq C \frac{1}{N^2} \|f^\vartheta\|_\infty^2.$$

So we have

$$\left\| 1_{|v|<N} w^\vartheta \mu^{1/2} \left[\partial_{ij} \phi^{ij} * w^{-\vartheta} \mu^{1/2} f^\vartheta \right] \right\|_2 \leq CN^2 \|f^\vartheta\|_2 + \frac{C}{N} \|f^\vartheta\|_\infty. \quad (2.19)$$

In a similar manner,

$$\left\| 1_{|v|<N} w^\vartheta \mu^{1/2} \left[\partial_i \phi^{ij} * (v_j w^{-\vartheta} \mu^{1/2} f^\vartheta) \right] \right\|_2 \leq CN^2 \|f^\vartheta\|_2 + \frac{C}{N} \|f^\vartheta\|_\infty \quad (2.20)$$

and

$$\left\| 1_{|v|<N} v_i w^\vartheta \mu \left[\phi^{ij} * (v_j w^{-\vartheta} \mu^{1/2} f^\vartheta) \right] \right\|_2 \leq CN \|f^\vartheta\|_2 + \frac{C}{N^2} \|f^\vartheta\|_\infty. \quad (2.21)$$

Note that

$$\begin{aligned} w^\vartheta \mu^{1/2} \left[\partial_{ij} \phi^{ij} * (w^{-\vartheta} \mu^{1/2} f^\vartheta) \right] &= w^\vartheta \mu^{1/2} \left[-8\pi w^{-\vartheta} \mu^{1/2} f^\vartheta \right] \\ &= -8\pi \mu f^\vartheta. \end{aligned}$$

Thus,

$$\left\| 1_{|v|<N} w^\vartheta \mu^{1/2} \left[\partial_{ij} \phi^{ij} * (w^{-\vartheta} \mu^{1/2} f^\vartheta) \right] \right\|_2 \leq C \|f^\vartheta\|_2. \quad (2.22)$$

From (2.18) - (2.22), we have (2.17)

So the proof is complete. \square

3. MAXIMUM PRINCIPLE

In this section, we first define a weak solution for (1.28) and obtain the well-posedness and the maximum principle of the weak solution for (1.28). Due to the lack of regularity, we cannot use a direct contradiction argument for the weak solution as in the case of strong solutions. Therefore, we first construct a smooth approximated solution and then pass to the limit to obtain the maximum principle for the weak solution.

Definition 3.1. Let $h(t, x, v) \in L^\infty((0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3, w^\vartheta(v) dt dx dv)$ be a periodic function in $x \in \mathbb{T}^3 = [-\pi, \pi]^3$ satisfying

$$\int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (\sigma^{ij} \partial_i h \partial_j h)(s, x, v) dx dv ds < \infty,$$

where σ is defined as in (1.13). We say that h is a weak solution of (1.28), with $h(0) = h_0$ on $(0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3$ if for all $t \in (0, \infty)$ and all $\varphi \in C_{t,x,v}^{1,1,1}((0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3)$ such that $\varphi(t, x, v)$ is a periodic function in $x \in \mathbb{T}^3 = [-\pi, \pi]^3$ and $\varphi(t, x, \cdot)$ is compactly supported in \mathbb{R}^3 , it satisfies

$$\begin{aligned} & \iint_{\mathbb{T}^3 \times \mathbb{R}^3} h(t, x, v) \varphi(t, x, v) dx dv - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} h_0(x, v) \varphi(0, x, v) dx dv \\ &= \iiint_{(0,t) \times \mathbb{T}^3 \times \mathbb{R}^3} h(s, x, v) \left(\partial_s \varphi + v \cdot \nabla_x \varphi - \left(a_g + 2 \frac{\nabla w^\vartheta}{w^\vartheta} \sigma_G \right) \cdot \nabla_v \varphi \right) (s, x, v) \\ & \quad - \nabla_v h(s, x, v) \cdot (\sigma_G \nabla_v \varphi)(s, x, v) ds dx dv, \end{aligned} \quad (3.1)$$

where σ_G is defined as in (1.12) with $G = \mu + \mu^{1/2} g$.

Lemma 3.2. Assume (2.4). Let σ_G be the matrix defined as in (1.12) with $G = \mu + \mu^{1/2} g$. Let $\vartheta \in \mathbb{N} \cup \{0\}$, $\delta \geq 0$, and h be a classical solution of (1.28). Then there exist $C = C(\vartheta)$, $0 < \varepsilon \ll 1$ such that if $\|g\|_\infty < \varepsilon$, then

$$\sup_{0 \leq s \leq t} \|h(s)\|_{L^2}^2 + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (\sigma^{ij} \partial_i h \partial_j h)(s, x, v) dx dv ds \leq C(t) \|h(0)\|_{L^2}^2. \quad (3.2)$$

Proof. Multiplying (1.28) by h and integrating both sides of the resulting equation, we have

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{1}{2} (h^2(t, x, v) - h^2(0, x, v)) dx dv = \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (\bar{A}_g^\vartheta h(s, x, v)) h(s, x, v) dx dv ds. \quad (3.3)$$

By Lemma 2.4, we have

$$C^{-1} \sigma^{ij} \partial_i h \partial_j h \leq \sigma_G^{ij} \partial_i h \partial_j h \leq C \sigma^{ij} \partial_i h \partial_j h, \quad (3.4)$$

for some C . By Lemma 2.4 and the Young inequality, we have

$$\begin{aligned} \left| \frac{\partial_i w^\vartheta}{w^\vartheta} \sigma_G^{ij} (\partial_j h) h \right| &\leq C(1 + |v|)^{-1} (\sigma_G^{ij} \partial_i h \partial_j h)^{1/2} (\sigma_G^{ij} h^2)^{1/2} \\ &\leq \varepsilon \sigma_G^{ij} \partial_i h \partial_j h + C_\varepsilon (1 + |v|)^{-1} \sigma_G^{ij} h^2 \\ &\leq \varepsilon \sigma^{ij} \partial_i h \partial_j h + C_\varepsilon h^2. \end{aligned} \quad (3.5)$$

In a similar manner, by (2.5) and the Young inequality, we have

$$\begin{aligned} \left| \left\{ \phi * [v_i \mu^{1/2} g] \right\} (\partial_i h) h \right| &\leq C \|g\|_\infty D_\mu(\nabla_v h; v)^{1/2} (\sigma^{ij} h^2)^{1/2} \\ &\leq \varepsilon \sigma^{ij} \partial_i h \partial_j h + \varepsilon h^2 \end{aligned} \quad (3.6)$$

and

$$\left| \left\{ \phi^{ij} * [\mu^{1/2} \partial_j g] \right\} \partial_i h h \right| \leq \varepsilon \sigma^{ij} \partial_i h \partial_j h + \varepsilon h^2. \quad (3.7)$$

Thus from (3.3) - (3.7), we have

$$\begin{aligned} &\iint_{\mathbb{T}^3 \times \mathbb{R}^3} h^2(t, x, v) dx dv + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (\sigma^{ij} \partial_i h \partial_j h)(s, x, v) dx dv ds \\ &\leq \iint_{\mathbb{T}^3 \times \mathbb{R}^3} h^2(0, x, v) dx dv + \varepsilon \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (\sigma^{ij} \partial_i h \partial_j h)(s, x, v) dx dv ds + C_\varepsilon \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} h^2(s, x, v) dx dv ds. \end{aligned}$$

Absorbing the second term of the RHS to the LHS and applying the Gronwall inequality to the resulting equation, we have (3.2). \square

Lemma 3.3. *Assume (2.4). Then there exists a unique weak solution to (1.28) which satisfies (3.2)*

Proof. We approximate g by $g^\delta \in C^\infty$ and h_0 by $h_0^\delta \in C^\infty$ such that $\|g^\delta\|_\infty \leq \|g\|_\infty$, $\|h_0^\delta\|_\infty \leq \|h_0\|_\infty$ and

$$\lim \|g^\delta - g\|_\infty = 0, \quad \lim \|h_0^\delta - h_0\|_1 = 0.$$

Consider

$$\partial_t h^\delta + v \cdot \nabla_x h^\delta = \bar{A}_{g^\delta}^\vartheta h^\delta \quad (3.8)$$

$$h^\delta(0, x, v) = h_0^\delta(x, v).$$

By Lemma 2.4, $\sigma_G \geq 0$. Since $\sigma_G \geq 0$, it is rather standard (for instance, by adding regularization $\varepsilon(\nabla_{x,v})^{2m}$, for some large integer m , then letting $\varepsilon \rightarrow 0$, if necessary) that there exists a solution h^δ to the linear equation (3.8). Since g^δ and h_0^δ are smooth, we can derive a similar energy estimate for the derivatives of h^δ by taking derivatives of the above equation and multiplying by the derivatives of h^δ and integrating both sides of the resulting equation as in [13]. For more details, see [13]. Therefore, h^δ is smooth.

By (3.2), $\|h^\delta(s)\|_{L^2}^2$ is uniformly bounded on $0 \leq s \leq t$. Therefore, there exists h such that $h^\delta \rightarrow h$ weakly in L^2 . Multiplying (3.8) by a test function φ , integrating both sides of the resulting equation, and taking the integration by parts, we have

$$\begin{aligned} & \iint_{\mathbb{T}^3 \times \mathbb{R}^3} h^\delta(t, x, v) \varphi(t, x, v) dx dv - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} h_0^\delta(x, v) \varphi(0, x, v) dx dv \\ &= \iiint_{(0, t) \times \mathbb{T}^3 \times \mathbb{R}^3} h^\delta(s, x, v) \left(\partial_s \varphi(s, x, v) + v \cdot \nabla_x \varphi(s, x, v) - \left(a_{g^\delta} + 2 \frac{\nabla w^\vartheta}{w^\vartheta} \sigma_{G^\delta} \right) \cdot \nabla_v \varphi \right. \\ & \quad \left. + \nabla_v \cdot (\sigma_{G^\delta} \nabla_v \varphi) \right) ds dx dv, \end{aligned} \quad (3.9)$$

where $G^\delta = \mu + \sqrt{\mu} g^\delta$. Since $h^\delta \rightarrow h$ weakly in L^2 , taking $\delta \rightarrow 0$ in (3.9) we have (3.1). Therefore h is a weak solution of (1.28). The second assertion is an analogue of Lemma 3.2. Let h and \tilde{h} be weak solutions to (1.28). Then $h - \tilde{h}$ is also a weak solution to (1.28) with zero initial data. Therefore, we have $\sup_{0 \leq s \leq t} \|(h - \tilde{h})(s)\|_{L^2}^2 = 0$. Thus we obtain the uniqueness. \square

Before we derive the maximum principle for weak solutions, we establish the maximum principle for strong solutions. We first derive the maximum principle for strong solutions in bounded domains. The following technique is similar to that in [18].

Lemma 3.4. *Assume (2.4). Let $h \in C_{t,x,v}^{1,1,2}([0, T] \times \mathbb{T}^3 \times B(0; M))$ be a periodic function satisfying $\mathcal{M}_g^\vartheta h \leq 0$. Then h attains its maximum only at $t = 0$ or $|v| = M$.*

Proof. Let us assume that $\max_{(t,x,v) \in [0,T] \times \mathbb{T}^3 \times B(0;M)} h(t, x, v) > 0$ and $\mathcal{M}_g^\vartheta h < 0$. Suppose that h attains its maximum at an interior point $(t, x, v) \in [0, T] \times \mathbb{T}^3 \times B(0; M)$ or at (T, x, v) with (x, v) lying in the interior. Since $\sigma_G \geq 0$ by Lemma 2.4, we have $\partial_t h \geq 0$, $\nabla_x h = 0$, and $\nabla_v h = 0$ while $\sigma_G^{ij} \partial_{ij} h \leq 0$ and $h(t, x, v) > 0$. Thus $\mathcal{M}_g^\vartheta h(t, x, v) \geq 0$ and this gives a contradiction. Suppose h attains its maximum at $|x| = \pi$. Since h is periodic in x , we can assume that h attains its maximum at $x = \pi$, $v \geq 0$ or $x = -\pi$, $v \leq 0$. Then $\partial_t h = 0$, $v \cdot \nabla_x h \geq 0$, and $\nabla_v h = 0$ while $\sigma_G^{ij} \partial_{ij} h \leq 0$ and $h(t, x, v) > 0$. Thus $\mathcal{M}_g^\vartheta h(t, x, v) \geq 0$ which makes a contradiction too.

In the case of $\mathcal{M}_g^\vartheta h \leq 0$, define $h^k := h - kt$ for $k > 0$, then $\mathcal{M}_g^\vartheta h^k < 0$. Thus we have

$$\sup_{(t,x,v) \in [0,T] \times \mathbb{T}^3 \times B(0;M)} h^k(t, x, v) = \sup_{t=0 \text{ or } |v|=M} h^k(t, x, v).$$

Taking $k \rightarrow 0$, we complete the proof. \square

Lemma 3.5. *Assume (2.4). There exists $\varphi \in C^{1,1,2}([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$ with $\varphi \geq 0$, which satisfies $\mathcal{M}_g^\vartheta \varphi \geq 0$ and $\varphi \rightarrow \infty$ as $|v| \rightarrow \infty$ uniformly in $t \in [0, T]$.*

Proof. Define,

$$\varphi(t, x, v) := \varphi(t, v) = \alpha_1(t) + \alpha_2(t)|v|^2. \quad (3.10)$$

Then

$$\begin{aligned}
\mathcal{M}_g^\vartheta \varphi &= \alpha'_1(t) + \alpha'_2(t)|v|^2 - 2\alpha_2(t)\nabla_v \cdot (\sigma_G v) \\
&\quad - 2\alpha_2(t)a_g \cdot v - 2\alpha_2(t)\frac{\partial_i w^\vartheta}{w^\vartheta}\sigma^{ij}v_j \\
&= \alpha'_1(t) + \alpha'_2(t)|v|^2 - 2\alpha_2(t)\partial_i \sigma_G^{ij}v_j - 2\alpha_2(t)\sigma_G^{ii} \\
&\quad - 2\alpha_2(t)a_g \cdot v - 2\alpha_2(t)\frac{\partial_i w^\vartheta}{w^\vartheta}\sigma^{ij}v_j.
\end{aligned}$$

Note that

$$\begin{aligned}
|\partial_i \sigma_G^{ij}| &\leq C\|g\|_\infty(1+|v|)^{-2}, \quad |\sigma_G^{ii}| \leq C\|g\|_\infty(1+|v|)^{-1}, \\
|a_g| &\leq C\|g\|_\infty(1+|v|)^{-1}, \quad \text{and} \quad \frac{\partial_i w^\vartheta}{w^\vartheta}\sigma^{ij}v_j \leq C\|g\|_\infty(1+|v|)^{-3}.
\end{aligned}$$

Choose $\alpha_1(t) = \alpha_2(t) := \exp(kt)$. Then $\varphi \geq 0$. Moreover,

$$\mathcal{M}_g^\vartheta \varphi \geq e^{kt} (k(1+|v|)^2 - C(1+|v|)^{-1} - C(1+|v|)^{-1}(1+|v|)) \geq 0$$

for a sufficiently large k . □

Lemma 3.6. Assume (2.4). Let $h \in C_{t,x,v}^{1,1,2}([0, T] \times \mathbb{T}^3 \times \mathbb{R}^3)$ be a periodic function satisfying $\mathcal{M}_g^\vartheta h \leq 0$. Then h attains its maximum only at $t = 0$.

Proof. Fix $\lambda > 0$. Let φ be a barrier function obtained in Lemma 3.5. Define $\eta^\lambda(t, x, v) := h(t, x, v) - \lambda\varphi(t, x, v)$, then $\mathcal{M}_g^\vartheta \eta^\lambda \leq 0$. Thus we can apply Lemma 3.4 on the domain $[0, T] \times \mathbb{T}^3 \times B(0; M)$. Then we have

$$\eta^\lambda(t, x, v) \leq \sup_{t=0 \text{ or } |v|=M} \eta^\lambda(t, x, v), \quad \text{for } (t, x, v) \in [0, T] \times \mathbb{T}^3 \times B(0; M).$$

Note that

$$\eta^\lambda(0, x, v) = h(0, x, v) - \lambda\varphi(0, x, v) \leq h(0, x, v) \leq \sup_{x,v} h(0, x, v).$$

For a sufficiently large M , we have

$$\eta^\lambda(t, x, v) = h(t, x, v) - \lambda\varphi(t, x, v) = h(t, x, v) - \lambda(\alpha_1(t) + \alpha_2(t)M^2) \leq \sup_{x,v} h(0, x, v)$$

for $|v| = M$. Thus

$$\eta^\lambda(t, x, v) \leq \sup_{x,v} h(0, x, v), \quad \text{for } (t, x, v) \in [0, T] \times \mathbb{T}^3 \times B(0; M).$$

Since M is an arbitrary large enough constant, we can take $M \rightarrow \infty$. Then we have

$$\eta^\lambda(t, x, v) \leq \sup_{x,v} h(0, x, v), \quad \text{for } (t, x, v) \in [0, T] \times \mathbb{T}^3 \times \mathbb{R}^3.$$

Taking $\lambda \rightarrow 0$, we have

$$h(t, x, v) \leq \sup_{x,v} h(0, x, v).$$

Thus we complete the proof. □

Now we will derive the maximum principle for weak solutions.

Lemma 3.7. *Assume (2.4) and $g \in C^0$ and $\|h_0\|_\infty < \infty$. Then the weak solution to (1.28) satisfies*

$$\sup_t \|h(t)\|_\infty \leq \|h_0\|_\infty \quad (3.11)$$

Proof. Approximating g by $g^\delta \in C^\infty$ and h_0 by $h_0^\delta \in C^\infty$ as in Lemma 3.3, we can obtain a smooth solution h^δ to (1.28). Thus by Lemma 3.6, we have

$$\sup_t \|h^\delta(t)\|_\infty \leq \|h_0^\delta\|_\infty \leq \|h_0\|_\infty.$$

In a similar manner to Lemma 3.2 we can derive an energy estimate for $h^\delta - h^{\delta'}$ and we can show that h^δ is a Cauchy sequence in L^2 . Therefore there exists h such that $\|h^\delta - h\|_{L^2} \rightarrow 0$. In a similar manner to Lemma 3.3, we can show that h is a weak solution. Since $\sup_t \|h^\delta(t)\|_\infty \leq \|h_0\|_\infty$ and $\|h^\delta - h\|_{L^2} \rightarrow 0$, we can obtain (3.11). \square

4. L^2 DECAY

In this section, we will establish a weighted L^2 estimate for (1.25). We will adapt techniques in [9], [13], and [24].

As a starting point, we prove that (1.25) has a unique weak solution globally in time.

Definition 4.1. Let $f(t, x, v) \in L^\infty((0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3, w^\vartheta(v) dt dx dv)$ be a periodic function in $x \in \mathbb{T}^3 = [-\pi, \pi]^3$ satisfying

$$\int_0^t \|f(s)\|_{\sigma, \vartheta}^2 ds < \infty. \quad (4.1)$$

We say that f is a weak solution of the Landau equation (1.6), (1.25) on $(0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3$ if for all $t \in (0, \infty)$ and all $\varphi \in C_{t,x,v}^{1,1,1}((0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3)$ such that $\varphi(t, x, v)$ is a periodic function in $x \in \mathbb{T}^3 = [-\pi, \pi]^3$ and $\varphi(t, x, \cdot)$ is compactly supported in \mathbb{R}^3 , it satisfies

$$\begin{aligned} & \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f(t, x, v) \varphi(t, x, v) dx dv - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} f_0(x, v) \varphi(0, x, v) dx dv \\ &= -(f, \varphi)_\sigma + \iiint_{(0,t) \times \mathbb{T}^3 \times \mathbb{R}^3} f(s, x, v) \left(\partial_s \varphi(s, x, v) + v \cdot \nabla_x \varphi(s, x, v) + a_g(s, x, v) \cdot \nabla_v \varphi(s, x, v) \right. \\ & \quad + K \varphi(s, x, v) + \partial_i \sigma^i(s, x, v) \varphi(s, x, v) - \partial_i \left\{ \phi^{ij} * [\mu^{1/2} \partial_j g] \right\} (s, x, v) \varphi(s, x, v) \\ & \quad \left. + \left\{ \phi^{ij} * [v_i \mu^{1/2} \partial_j g] \right\} (s, x, v) \varphi(s, x, v) \right) - \sigma_{\mu^{1/2} g}(s, x, v) \partial_i f(s, x, v) \partial_j \varphi(s, x, v) ds dx dv. \end{aligned} \quad (4.2)$$

Let $f(t) = U(t, s) f_0$ be a solution of the following equation

$$f_t + v \cdot \partial_x f = \bar{A}_g f, \quad (4.3)$$

$$f(s) = U(s, s) f_0 = f_0,$$

where \bar{A}_g is defined as in (1.15). Then by the Duhamel principle, the solution of (1.25) is

$$f(t) = U(t, 0) f_0 + \int_0^t U(t, \tau) \bar{K}_g f(\tau) d\tau. \quad (4.4)$$

Lemma 4.2. *Assume (2.4). Then there exists a unique weak solution f to (1.25) in the sense of Definition 4.1 with $f(0) = f_0$, which satisfies*

$$\sup_{0 \leq s \leq t} \|f(s)\|_\infty \leq C(t) \|f_0\|_\infty.$$

Sketch of proof. It is clear from (4.4) and by the Gronwall inequality. \square

For any real-valued function $f(v)$, we define the projection onto the $\text{span}\{\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}\}$ in $L^2(\mathbb{R}^3)$ as

$$\mathbf{P}f := \left(a_f(t, x) + v \cdot b_f(t, x) + \frac{(|v|^2 - 3)}{2} c_f(t, x) \right) \sqrt{\mu}, \quad (4.5)$$

where

$$\begin{aligned} a_f &:= \frac{\langle f, \sqrt{\mu} \rangle}{|\langle \sqrt{\mu}, \sqrt{\mu} \rangle|^2}, \\ b_f^i &:= \frac{\langle f, v_i \sqrt{\mu} \rangle}{|\langle v_i \sqrt{\mu}, v_i \sqrt{\mu} \rangle|^2}, \\ c_f &:= \frac{\langle f, (|v|^2 - 3)\sqrt{\mu}/2 \rangle}{|\langle (|v|^2 - 3)\sqrt{\mu}/2, (|v|^2 - 3)\sqrt{\mu}/2 \rangle|^2}. \end{aligned}$$

We will prove the positivity of L . By Lemma 2.6, L is only semi-positive;

$$(Lf, f) \geq C \|(I - \mathbf{P})f\|_\sigma^2.$$

Now we will estimate $\mathbf{P}f$ in terms of $(I - \mathbf{P})f$. The following lemma is an adaptation of Lemma 6.1 in [9].

Lemma 4.3 (Lemma 6.1 in [9]). *Assume (2.4). Let f be a weak solution of (1.6), (1.7), (1.25). Then there exist C and a function $\eta(t) \leq C\|f(t)\|_2^2$, such that*

$$\int_s^t \|\mathbf{P}f(\tau)\|_\sigma^2 \leq \eta(t) - \eta(s) + C \int_s^t \|(I - \mathbf{P})f(\tau)\|_\sigma^2.$$

Proof. For every periodic test function ψ , f satisfies,

$$\begin{aligned} & - \int_s^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} v \cdot \nabla_x \psi f - \int_s^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \partial_t \psi f \\ &= - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(t) + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(s) + \int_s^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} -\psi L(I - \mathbf{P})f + \psi \Gamma(f, f). \end{aligned} \quad (4.6)$$

By convention, we denote $a(t, x) = a_f(t, x)$, $b(t, x) = b_f(t, x)$, and $c(t, x) = c_f(t, x)$, where a_f , b_f , and c_f are defined as in (4.5). We note that, with such choices $\eta(t) = - \int \int_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(t) dx dv$, and

$\psi = p(v)\phi(t, x)$ for some $|p(v)| \leq \exp(-|v|^2/4)$ and $\|\phi(t)\|_2 \leq C(\|a(t)\|_2 + \|b(t)\|_2 + \|c(t)\|_2)$. Thus,

$$\begin{aligned} |\eta(t)| &\leq \|f(t)\|_2 \left(\int_{\mathbb{T}^3 \times \mathbb{R}^3} p(v) |\phi(t, x)|^2 dx dv \right)^{1/2} \\ &= C \|f(t)\|_\infty \left(\int_{\mathbb{T}^3} |\phi(t, x)|^2 dx \right)^{1/2} \\ &\leq C \|f(t)\|_2 \|\phi(t)\|_2 \\ &\leq C \|f(t)\|_2^2. \end{aligned}$$

Without loss of generality, we can take $s = 0$.

Step 1. Estimate of $\nabla_x \Delta^{-1} \partial_t a = \nabla_x \partial_t \phi_a$. Choosing a test function $\psi = \phi \sqrt{\mu}$ with ϕ dependent only on x , we have (Note that $\int \sqrt{\mu} L f = \int \sqrt{\mu} \Gamma(f, f) = 0$)

$$\sqrt{2\pi} \int_{\mathbb{T}^3} [a(t + \varepsilon) - a(t)] \phi(x) = 2\pi \sqrt{2\pi} \int_t^{t+\varepsilon} \int_{\mathbb{T}^3} (b \cdot \nabla_x) \phi(x).$$

Therefore,

$$\int_{\mathbb{T}^3} \phi \partial_t a = \sqrt{2\pi} \int_{\mathbb{T}^3} (b \cdot \nabla_x) \phi.$$

First, take $\phi = 1$. Then, we have $\int_{\mathbb{T}^3} \partial_t a(t) dx = 0$ for all $t > 0$. On the other hand, for all $\phi(x) \in H^1(\mathbb{T}^3)$, we have

$$\left| \int_{\mathbb{T}^3} \phi(x) \partial_t a dx \right| \lesssim \|b\|_2 \|\phi\|_{H^1}.$$

Therefore, for all $t > 0$, $\|\partial_t a(t)\|_{(H^1)^*} \lesssim \|b(t)\|_2$. Since $\int_{\mathbb{T}^3} \partial_t a dx = 0$ for all $t > 0$, we can find a solution of the Poisson equation with the Neumann boundary condition $-\Delta \Phi_a = \partial_t a(t)$, $\frac{\partial \Phi_a}{\partial n} = 0$ at $\partial \mathbb{T}^3$. Let ϕ_a be a solution of the Poisson equation with the Neumann boundary condition $-\Delta \phi_a = a(t)$, $\frac{\partial \phi_a}{\partial n} = 0$ at $\partial \mathbb{T}^3$. Then $\Phi_a = \partial_t \phi_a$. Moreover, we have

$$\|\nabla_x \partial_t \phi_a\|_2 = \|\Phi_a\|_{H^1} \lesssim \|\partial_t a(t)\|_{(H^1)^*} \lesssim \|b(t)\|_2. \quad (4.7)$$

Step 2. Estimate of $\nabla_x \Delta^{-1} \partial_t b^j = \nabla_x \partial_t \phi_b^j$. Choosing a test function $\psi = \phi(x) v_i \sqrt{\mu}$, we have

$$\begin{aligned} &2\pi \sqrt{2\pi} \int_{\mathbb{T}^3} [b_i(t + \varepsilon) - b_i(t)] \phi \\ &= 2\pi \sqrt{2\pi} \int_t^{t+\varepsilon} \int_{\mathbb{T}^3} \partial_i \phi [a + c] + \int_t^{t+\varepsilon} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \sum_{j=1}^d v_j v_i \sqrt{\mu} \partial_j \phi (I - \mathbf{P}) f \\ &\quad + \int_t^{t+\varepsilon} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \phi v_i \Gamma(f, f) \sqrt{\mu}. \end{aligned}$$

Therefore,

$$\int_{\mathbb{T}^3} \partial_t b_i(t) \phi = \int_{\mathbb{T}^3} \partial_i \phi [a(t) + c(t)] + \frac{1}{2\pi \sqrt{2\pi}} \left\{ \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \sum_{j=1}^d v_j v_i \sqrt{\mu} \partial_j \phi (I - \mathbf{P}) f(t) \right.$$

$$+ \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \phi v_i \Gamma(f, f)(t) \sqrt{\mu} \Big\}.$$

By the Hölder inequality and Theorem 2.8

$$\begin{aligned} & \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \sum_{j=1}^d v_i v_j \sqrt{\mu} \partial_j \phi (I - \mathbf{P}) f(t) \\ & \leq C \|(1 + |v|)^{-1/2} (I - \mathbf{P}) f(t)\|_2^2 \left\| (1 + |v|)^{-\frac{-1}{2}} \sum_{j=1}^d v_i v_j \sqrt{\mu} \partial_j \phi \right\|_2 \\ & \leq C \|(I - \mathbf{P})\|_\sigma \|\phi\|_2 \end{aligned}$$

and

$$\left\| \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \phi v_i \Gamma(f, f)(t) \sqrt{\mu} \right\| \leq C \|f\|_\infty \|f\|_\sigma \|\phi v_i \mu\|_\sigma \leq \|f\|_\infty \|f\|_\sigma \|\phi\|_2.$$

For fixed $t > 0$, we choose $\phi = \Phi_b^i$, where Φ_b^i is a solution of the Poisson equation with the Dirichlet boundary condition $-\Delta \Phi_b^i = \partial_t b_i(t)$, $\Phi_b^i|_{\partial \mathbb{T}^3} = 0$. Let ϕ_b^i be a solution of the Poisson equation with the Dirichlet boundary condition $-\Delta \phi_b^i = b_i(t)$, $\phi_b^i|_{\partial \mathbb{T}^3} = 0$. Then $\Phi_b^i = \partial_t \phi_b^i$. By the Poincaré inequality,

$$\begin{aligned} \int_{\mathbb{T}^3} |\nabla_x \partial_t \phi_b^i(t)|^2 dx &= \int_{\mathbb{T}^3} |\nabla_x \Phi_b^i|^2 dx = - \int_{\mathbb{T}^3} \Delta_x \Phi_b^i \Phi_b^i dx \\ &\leq \varepsilon \{ \|\nabla_x \Phi_b^i\|_2^2 + \|\Phi_b^i\|_2^2 \} \\ &\quad + C_\varepsilon \{ \|a(t)\|_2^2 + \|c(t)\|_2^2 + \|(I - \mathbf{P})f(t)\|_\sigma^2 + \|f(t)\|_\infty^2 \|f(t)\|_\sigma^2 \} \\ &\leq C\varepsilon \|\nabla_x \Phi_b^i\|_2^2 + C_\varepsilon \{ \|a(t)\|_2^2 + \|c(t)\|_2^2 + \|(I - \mathbf{P})f(t)\|_\sigma^2 + \|f(t)\|_\infty^2 \|f(t)\|_\sigma^2 \}, \end{aligned}$$

for every $\varepsilon > 0$. Now, we choose small ε , such that $C\varepsilon \leq 1/4$. Then we can absorb the first term in RHS to the LHS. Then we have for all $t > 0$,

$$\|\nabla_x \partial_t \phi_b^i(t)\|_2 \leq C_\varepsilon \{ \|a(t)\|_2 + \|c(t)\|_2 + \|(I - \mathbf{P})f(t)\|_\sigma + \|f(t)\|_\infty \|f(t)\|_\sigma \}. \quad (4.8)$$

Step 3. Estimate of $\nabla_x \Delta^{-1} \partial_t c = \nabla_x \partial_t c$. Choosing a test function $\psi = \phi(x) \left(\frac{|v|^2 - 3}{2} \right) \sqrt{\mu}$, we have

$$\begin{aligned} & 3\pi\sqrt{2\pi} \int_{\mathbb{T}^3} \phi(x) [c(t + \varepsilon) - c(t)] \\ &= 2\pi\sqrt{2\pi} \int_t^{t+\varepsilon} \int_{\mathbb{T}^3} b \cdot \nabla_x \phi - \int_t^{t+\varepsilon} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (I - \mathbf{P}) f \left(\frac{|v|^2 - 3}{2} \right) \sqrt{\mu} (v \cdot \nabla_x) \phi \\ &\quad + \int_t^{t+\varepsilon} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \phi \Gamma(f, f) \left(\frac{|v|^2 - 3}{2} \right) \sqrt{\mu}. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \int_{\mathbb{T}^3} \phi(x) \partial_t c(t) \\
&= \frac{2}{3} \int_{\mathbb{T}^3} b(t) \cdot \nabla_x \phi + \frac{1}{3\pi\sqrt{2\pi}} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (I - \mathbf{P})f(t) \left(\frac{|v|^2 - 3}{2} \right) \sqrt{\mu}(v \cdot \nabla_x) \phi \\
&+ \frac{1}{3\pi\sqrt{2\pi}} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \phi \Gamma(f, f)(t) \left(\frac{|v|^2 - 3}{2} \right) \sqrt{\mu}.
\end{aligned}$$

Similarly to step 2,

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} (I - \mathbf{P})f(t) \left(\frac{|v|^2 - 3}{2} \right) \sqrt{\mu}(v \cdot \nabla_x) \phi \leq C \|(I - \mathbf{P})\|_\sigma \|\nabla_x \phi\|_2$$

and

$$\left\| \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \phi \Gamma(f, f)(t) \left(\frac{|v|^2 - 3}{2} \right) \sqrt{\mu} \right\| \leq \|f\|_\infty \|f\|_\sigma \|\phi\|_2.$$

For fixed $t > 0$, we choose $\phi = \Phi_c$, where Φ_c is a solution of the Poisson equation with the Dirichlet boundary condition $-\Delta \Phi_c = \partial_t c(t)$, $\Phi_c|_{\partial \mathbb{T}^3} = 0$. Let ϕ_c be a solution of the Poisson equation with the Dirichlet boundary condition $-\Delta \phi_c = c(t)$, $\phi_c(t)|_{\partial \mathbb{T}^3} = 0$. Then $\Phi_c = \partial_t \phi_c$. By the Poincaré inequality,

$$\begin{aligned}
\int_{\mathbb{T}^3} |\nabla_x \partial_t \phi_c(t)|^2 dx &= \int_{\mathbb{T}^3} |\nabla_x \Phi_c|^2 dx = - \int_{\mathbb{T}^3} \Delta_x \Phi_c \Phi_c dx \\
&\leq \varepsilon \{ \|\nabla_x \Phi_c\|_2^2 + \|\Phi_c\|_2^2 \} \\
&+ C_\varepsilon \{ \|b(t)\|_2^2 + \|(I - \mathbf{P})f(t)\|_\sigma^2 + \|f(t)\|_\infty^2 \|f(t)\|_\sigma^2 \} \\
&\leq C\varepsilon \|\nabla_x \Phi_c\|_2^2 + C_\varepsilon \{ \|b(t)\|_2^2 + \|(I - \mathbf{P})f(t)\|_\sigma^2 + \|f(t)\|_\infty^2 \|f(t)\|_\sigma^2 \}.
\end{aligned}$$

Therefore, for all $t > 0$,

$$\|\nabla_x \partial_t \phi_c(t)\|_2 \leq C_\varepsilon \{ \|b(t)\|_2 + \|(I - \mathbf{P})f(t)\|_\sigma + \|f(t)\|_\infty \|f(t)\|_\sigma \}. \quad (4.9)$$

Step 4. Estimate of c . Choosing a test function $\psi = (|v|^2 - 5)\sqrt{\mu}v \cdot \nabla_x \phi_c$, we have

$$\begin{aligned}
-10\pi\sqrt{2\pi} \int_0^t \int_{\mathbb{T}^3} \Delta_x \phi_c c &= - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(t) + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(0) \\
&+ \sum_{i=1}^d \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 5) v_i \sqrt{\mu} \partial_t \partial_i \phi_c f \\
&- \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi L(I - \mathbf{P})f + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi \Gamma(f, f). \\
&= - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(t) + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(0) \\
&+ \sum_{i,j=1}^d \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 5) v_i v_j \mu \partial_t \partial_i \phi_c b_j \\
&+ \sum_{i=1}^d \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 5) v_i \sqrt{\mu} \partial_t \partial_i \phi_c (I - \mathbf{P})f \\
&- \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi L(I - \mathbf{P})f + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi \Gamma(f, f).
\end{aligned}$$

Note that $\int (|v|^2 - 5) v_i v_j \mu = 0$. Therefore, the third term of RHS is zero. Moreover,

$$\begin{aligned}
&\sum_{i=1}^d \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 5) v_i \sqrt{\mu} \partial_t \partial_i \phi_c (I - \mathbf{P})f \\
&\leq C \|\nabla_x \partial_t \phi_c\|_2 \|(I - \mathbf{P})f\|_\sigma \\
&\leq C(C_\varepsilon \{\|b\|_2 + \|(I - \mathbf{P})f\|_\sigma + \|f\|_\infty \|f\|_\sigma\}) \|(I - \mathbf{P})f\|_\sigma \\
&\leq \varepsilon \|b\|_2^2 + C_\varepsilon \{\|(I - \mathbf{P})f\|_\sigma^2 + \|f\|_\infty^2 \|f\|_\sigma^2\}, \\
&\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi L(I - \mathbf{P})f = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} L\psi (I - \mathbf{P})f \\
&\leq C \|\nabla_x \phi_c\|_2 \|(I - \mathbf{P})f\|_\sigma \\
&\leq C \|c\|_2 \|(I - \mathbf{P})f\|_\sigma \\
&\leq \varepsilon \|c\|_2^2 + C_\varepsilon \|(I - \mathbf{P})f\|_\sigma^2
\end{aligned}$$

and

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi \Gamma(f, f) \leq C \|f\|_\infty \|f\|_\sigma \|c\|_2 \leq \varepsilon \|c\|_2^2 + C_\varepsilon \|f\|_\infty^2 \|f\|_\sigma^2.$$

For a small $\varepsilon > 0$, we can absorb $\|c\|_2^2$ on the RHS to the LHS. By (4.9), we have

$$\int_0^t \|c(s)\|_2^2 ds \leq C(\eta(t) - \eta(0)) + \int_0^t C_\varepsilon \{\|(I - \mathbf{P})f\|_\sigma^2 + \|f\|_\infty^2 \|f\|_\sigma^2\} + \varepsilon \|b\|_2^2 ds. \quad (4.10)$$

Step 5. Estimate of b . We will estimate $(\partial_{ij} \phi_b^j) b_i$ for all $i, j = 1, \dots, d$, and $(\partial_{jj} \phi_b^i) b_i$ for $i \neq j$.

We first estimate $(\partial_{ij}\phi_b^j)b_i$. Choosing a test function $\psi = [(v_i)^2 - 1]\sqrt{\mu}\partial_j\phi_b^j$, we have

$$\begin{aligned} & - \sum_l \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} v_* [(v_i)^2 - 1] \sqrt{\mu} \partial_{lj} \phi_b^j f - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (v_i^2 - 1) \sqrt{\mu} \partial_t \partial_j \phi_b^j f \\ & = - \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(t) + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(0) \\ & \quad - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi L(I - \mathbf{P})f + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi \Gamma(f, f). \end{aligned} \quad (4.11)$$

Note that for $i \neq k$

$$\int [(v_i)^2 - 1] \mu = \int [(v_i)^2 - 1] (v_k)^2 \mu = 0,$$

and

$$\int [(v_i)^2 - 1] (v_i)^2 \mu = 2\sqrt{2\pi}.$$

Therefore,

$$\begin{aligned} & \sum_l \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} v_* [(v_i)^2 - 1] \sqrt{\mu} \partial_{lj} \phi_b^j f \\ & = \sum_l \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (v_*)^2 [(v_i)^2 - 1] \mu \partial_{lj} \phi_b^j b_l \\ & \quad + \sum_l \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} v_* [(v_i)^2 - 1] \sqrt{\mu} \partial_{lj} \phi_b^j (I - \mathbf{P})f \\ & = 2\sqrt{2\pi} \int_{\mathbb{T}^3} \partial_{ij} \phi_b^j b_i + \sum_l \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} v_* [(v_i)^2 - 1] \sqrt{\mu} \partial_{lj} \phi_b^j (I - \mathbf{P})f, \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} \left| \sum_l \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} v_* [(v_i)^2 - 1] \sqrt{\mu} \partial_{lj} \phi_b^j (I - \mathbf{P})f \right| & \leq C \int_0^t \|b\|_2 \|(I - \mathbf{P})f\|_\sigma \\ & \leq \varepsilon \|b\|_2^2 + C_\varepsilon \|(I - \mathbf{P})f\|_\sigma^2. \end{aligned} \quad (4.13)$$

Moreover

$$\begin{aligned} \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (v_i^2 - 1) \sqrt{\mu} \partial_t \partial_j \phi_b^j f & = \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (v_i^2 - 1) \mu \partial_t \partial_j \phi_b^j \frac{|v|^2 - 3}{2} c \\ & \quad + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (v_i^2 - 1) \sqrt{\mu} \partial_t \partial_j \phi_b^j (I - \mathbf{P})f. \end{aligned}$$

By (4.8),

$$\begin{aligned} \left| \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (v_i^2 - 1) \sqrt{\mu} \partial_t \partial_j \phi_b^j f \right| &\leq \int_0^t C_\varepsilon \{ \|a\|_2 + \|c\|_2 + \|(I - \mathbf{P})f\|_\sigma + \|f\|_\sigma \|f\|_\sigma \} \\ &\quad \times \{ \|c\|_2 + C_\vartheta \|(I - \mathbf{P})f\|_\sigma \} \\ &\leq \int_0^t C_\varepsilon \{ \|(I - \mathbf{P})f\|_\sigma^2 + \|f\|_\infty^2 \|f\|_\sigma^2 + \|c\|_2^2 \} + \varepsilon \|a\|_2^2. \end{aligned} \quad (4.14)$$

In a similar way to step 4,

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi L(I - \mathbf{P})f \leq \varepsilon \|b\|_2^2 + C_\varepsilon \|(I - \mathbf{P})f\|_\sigma^2. \quad (4.15)$$

and

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi \Gamma(f, f) \leq \varepsilon \|b\|_2^2 + C_\varepsilon \|f\|_\infty^2 \|f\|_\sigma^2. \quad (4.16)$$

Combining (4.11) - (4.16),

$$\begin{aligned} &\int \partial_{ij} \phi_b^j b_i \\ &\leq C(\eta(t) - \eta(0)) + \int_0^t C_\varepsilon \{ \|(I - \mathbf{P})f\|_\sigma^2 + \|f\|_\infty^2 \|f\|_\sigma^2 + \|c\|_2^2 \} + \varepsilon \{ \|a\|_2^2 + \|b\|_2^2 \}. \end{aligned} \quad (4.17)$$

Now we estimate $(\partial_{jj} \phi_b^i) b_i$. Choose test function $\psi = |v|^2 v_i v_j \sqrt{\mu} \partial_j \phi_b^i$ for $i \neq j$. Then

$$\begin{aligned} &-\sum_l \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} v_* |v|^2 v_i v_j \sqrt{\mu} \partial_{lj} \phi_b^i f - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 v_i v_j \sqrt{\mu} \partial_t \partial_j \phi_b^i f \\ &= -\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(t) + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(0) - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi L(I - \mathbf{P})f + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi \Gamma(f, f). \end{aligned} \quad (4.18)$$

Note that

$$\begin{aligned} &\sum_l \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} v_* |v|^2 v_i v_j \sqrt{\mu} \partial_{lj} \phi_b^i f \\ &= \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 (v_i)^2 (v_j)^2 \sqrt{\mu} [\partial_{ij} \phi_b^i b_j + \partial_{jj} \phi_b^i b_i] \\ &\quad + \sum_l \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} v_* |v|^2 v_i v_j \sqrt{\mu} \partial_{lj} \phi_b^i (I - \mathbf{P})f. \end{aligned} \quad (4.19)$$

From (4.17),

$$\begin{aligned} &\left| \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 (v_i)^2 (v_j)^2 \sqrt{\mu} \partial_{ij} \phi_b^i b_j \right| \\ &\leq C(\eta(t) - \eta(0)) + \int_0^t C_\varepsilon \{ \|(I - \mathbf{P})f\|_{\sigma, \vartheta}^2 + \|f\|_\infty^2 \|f\|_{\sigma, \vartheta}^2 + \|c\|_2^2 \} + \varepsilon \{ \|a\|_2^2 + \|b\|_2^2 \}, \end{aligned} \quad (4.20)$$

and

$$\begin{aligned} \left| \sum_l \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} v_* |v|^2 v_i v_j \sqrt{\mu} \partial_{lj} \phi_b^i (I - \mathbf{P}) f \right| &\leq \int_0^t C \|b\|_2 \|(I - \mathbf{P})f\|_\sigma \\ &\leq \int_0^t \varepsilon \|b\|_2^2 + C_\varepsilon \|(I - \mathbf{P})f\|_\sigma^2. \end{aligned} \quad (4.21)$$

Moreover, by (4.8)

$$\begin{aligned} \left| \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 v_i v_j \sqrt{\mu} \partial_t \partial_j \phi_b^i f \right| &= \left| \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} |v|^2 v_i v_j \sqrt{\mu} \partial_t \partial_j \phi_b^i (I - \mathbf{P}) f \right| \\ &\leq \int_0^t C_\varepsilon \{ \|a\|_2 + \|c\|_2 + \|(I - \mathbf{P})f\|_\sigma + \|f\|_\infty \|f\|_\sigma \} \|(I - \mathbf{P})f\|_\sigma \\ &\leq \int_0^t C_\varepsilon \{ \|(I - \mathbf{P})f\|_\sigma^2 + \|f\|_\infty^2 \|f\|_\sigma^2 \} + \varepsilon \{ \|a\|_2^2 + \|c\|_2^2 \}. \end{aligned} \quad (4.22)$$

Similarly to (4.15) and (4.16),

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi L(I - \mathbf{P})f \leq \varepsilon \|b\|_2^2 + C_\varepsilon \|(I - \mathbf{P})f\|_\sigma^2. \quad (4.23)$$

and

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi \Gamma(f, f) \leq \varepsilon \|b\|_2^2 + C_\varepsilon \|f\|_\infty^2 \|f\|_\sigma^2. \quad (4.24)$$

Combining (4.18) - (4.24) yields

$$\begin{aligned} &\int \partial_{jj} \phi_b^i b_i \\ &\leq C(\eta(t) - \eta(0)) + \int_0^t C_\varepsilon \{ \|(I - \mathbf{P})f\|_\sigma^2 + \|f\|_\infty^2 \|f\|_\sigma^2 + \|c\|_2^2 \} + \varepsilon \{ \|a\|_2^2 + \|b\|_2^2 \}. \end{aligned} \quad (4.25)$$

From (4.17) and (4.25) for small ε , we can absorb $\|b\|_2^2$ term on RHS to the LHS. Then we can conclude that

$$\int_0^t \|b(s)\|^2 ds \leq C(\eta(t) - \eta(0)) + \int_0^t C_\varepsilon \{ \|(I - \mathbf{P})f\|_\sigma^2 + \|f\|_\infty^2 \|f\|_\sigma^2 + \|c\|_2^2 \} + \varepsilon \|a\|_2^2 ds. \quad (4.26)$$

Step 6. Estimate of a. Choosing a test function

$$\psi = (|v|^2 - 10)v \cdot \nabla_x \phi_a \sqrt{\mu},$$

we have

$$\begin{aligned} &-\int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 10)v_i v_j \partial_{ij} \phi_a \sqrt{\mu} f - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 10)v_i \partial_t \partial_i \phi_a \sqrt{\mu} f \\ &= -\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(t) + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi f(0) - \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi L(I - \mathbf{P})f + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi \Gamma(f, f). \end{aligned} \quad (4.27)$$

Note that

$$\int (|v|^2 - 10) \frac{|v|^2 - 3}{2} (v_i)^2 \mu = 0.$$

Therefore,

$$\begin{aligned} & \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 10) v_i v_j \partial_{ij} \phi_a \sqrt{\mu} f \\ &= \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 10) (v_i)^2 \mu \partial_{ii} \phi_a a + \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 10) v_i v_j \partial_{ij} \phi_a \sqrt{\mu} (I - \mathbf{P}) f \end{aligned} \quad (4.28)$$

and

$$\begin{aligned} \left| \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 10) v_i v_j \partial_{ij} \phi_a \sqrt{\mu} (I - \mathbf{P}) f \right| &\leq \int_0^t C \|a\|_2 \|(I - \mathbf{P}) f\|_\sigma \\ &\leq \int_0^t \varepsilon \|a\|_2^2 + C_\varepsilon \|(I - \mathbf{P}) f\|_\sigma^2. \end{aligned} \quad (4.29)$$

Moreover, by (4.7)

$$\begin{aligned} \left| \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 10) v_i \partial_t \partial_i \phi_a \sqrt{\mu} f \right| &\leq \left| \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 10) (v_i)^2 \mu \partial_t \partial_i \phi_a b_i \right| \\ &\quad + \left| \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (|v|^2 - 10) v_i \partial_t \partial_i \phi_a \sqrt{\mu} (I - \mathbf{P}) f \right| \\ &\leq \int_0^t C \|b\|_2 \{ \|b\|_2 + C \|(I - \mathbf{P}) f\|_\sigma \} \\ &\leq \int_0^t C \{ \|b\|_2^2 + \|(I - \mathbf{P}) f\|_\sigma^2 \}. \end{aligned} \quad (4.30)$$

Similarly to step 4 and 5, we have

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi L(I - \mathbf{P}) f \leq \varepsilon \|a\|_2^2 + C_\varepsilon \|(I - \mathbf{P}) f\|_\sigma^2 \quad (4.31)$$

and

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \psi \Gamma(f, f) \leq \varepsilon \|a\|_2^2 + C_\varepsilon \|f\|_\infty^2 \|f\|_\sigma^2. \quad (4.32)$$

Similarly, from (4.27) - (4.32) for a small ε , we can absorb $\|a\|_2^2$ on the RHS to the LHS. Then we have

$$\int_0^t \|a(s)\|_2^2 ds \leq C(\eta(t) - \eta(0)) + \int_0^t C_\varepsilon \{ \|(I - \mathbf{P}) f(s)\|_\sigma^2 + \|f(s)\|_\infty^2 \|f(s)\|_\sigma^2 + \|b(s)\|_2^2 \} ds. \quad (4.33)$$

Combining (4.10), (4.26), and (4.33), we have

$$\begin{aligned} \int_0^t \|\mathbf{P} f\|_\sigma^2 &\leq C(\eta(t) - \eta(0)) + \int_0^t C_\varepsilon \{ \|(I - \mathbf{P}) f(s)\|_\sigma^2 + \|f(s)\|_\infty^2 \|f(s)\|_\sigma^2 \} ds \\ &\quad + \int_0^t \varepsilon \|\mathbf{P} f(s)\|_\sigma^2 ds \\ &\leq C(\eta(t) - \eta(0)) + \int_0^t C_\varepsilon \{ \|(I - \mathbf{P}) f(s)\|_\sigma^2 + \|f(s)\|_\infty^2 \|(I - \mathbf{P}) f(s)\|_\sigma^2 \} ds \\ &\quad + \int_0^t (C_\varepsilon \|f(s)\|_\infty^2 + \varepsilon) \|\mathbf{P} f(s)\|_\sigma^2 ds. \end{aligned}$$

Note that $C_\varepsilon = C\varepsilon^{-1}$. Choosing $\varepsilon_0 = \varepsilon$, we have $C_\varepsilon \|f(s)\|_\infty^2 \leq C_\varepsilon \varepsilon^2 = C\varepsilon \leq 1/4$ so that $\|\mathbf{P}f\|_\sigma^2$ term on the RHS can be absorbed to the LHS. Thus we complete the proof. \square

Corollary 4.4. *Assume (2.4). Let $f(t, x, v)$ be a weak solution of (1.5) - (1.7) in the sense of Definition 4.1. Then there exist a constant $0 < \delta' \leq 1/4$ and a function $0 \leq \eta(t) \leq C\|f(t)\|_2^2$, such that*

$$\int_s^t (L[f(\tau)], f(\tau)) d\tau \geq \delta' \left(\int_s^t \|f(\tau)\|_\sigma^2 d\tau - \{\eta(t) - \eta(s)\} \right). \quad (4.34)$$

Proof. By Lemma 2.6 and Lemma 4.3,

$$\begin{aligned} \int_s^t (L[f(\tau)], f(\tau)) d\tau &\geq \delta \int_s^t \|(I - \mathbf{P})f(\tau)\|_\sigma^2 d\tau \\ &\geq \delta \frac{C}{1+C} \int_s^t \|(I - \mathbf{P})f(\tau)\|_\sigma^2 d\tau + \delta \frac{1}{1+C} \int_s^t \|(I - \mathbf{P})f(\tau)\|_\sigma^2 d\tau \\ &\geq \delta \frac{C}{1+C} \int_s^t \|(I - \mathbf{P})f(\tau)\|_\sigma^2 d\tau + \delta \frac{1}{1+C} C \left(\int_s^t \|\mathbf{P}f(\tau)\|_\sigma^2 d\tau - \{\eta(t) - \eta(s)\} \right) \\ &= \frac{C\delta}{1+C} \left(\int_0^t \|f(\tau)\|_\sigma^2 d\tau - \{\eta(t) - \eta(s)\} \right). \end{aligned}$$

\square

Remark 4.5. Note that in Lemma 2.6, we can take $\delta > 0$ sufficiently small. Therefore we can also take δ' small enough.

Now we will prove Theorem 1.4. The proof is a modification of Theorem 5.1 in [24].

Proof of Theorem 1.4. We will prove

$$\begin{aligned} \sum_{0 \leq \bar{\vartheta} \leq 2\vartheta} \left(\frac{C_{\bar{\vartheta}}^*}{2} \{\|f(t)\|_{2, \bar{\vartheta}/2}^2 - \|f(s)\|_{2, \bar{\vartheta}/2}^2\} + \delta_{\bar{\vartheta}, 2\vartheta} \int_s^t \|f(\tau)\|_{\sigma, \bar{\vartheta}/2}^2 d\tau \right) - \delta' \{\eta(t) - \eta(s)\} \\ \leq C_{\vartheta} \int_s^t \|g(\tau)\|_\infty \|f(\tau)\|_{\sigma, \vartheta}^2 d\tau \end{aligned} \quad (4.35)$$

by the induction on ϑ .

Basis step ($\vartheta = 0$). Multiplying (1.25) by f , integrating both sides of the resulting equation, by Theorem 2.8 and Corollary 4.4, we have

$$\frac{1}{2} \{\|f(t)\|_2^2 - \|f(s)\|_2^2\} + \delta' \left(\int_s^t \|f(\tau)\|_\sigma^2 d\tau - \{\eta(t) - \eta(s)\} \right) \leq C \int_s^t \|g(\tau)\|_\infty \|f(\tau)\|_\sigma^2 d\tau.$$

Inductive step. Suppose that (4.35) holds for $\vartheta - 1/2$. Multiplying (1.25) by $w^{2\vartheta} f$, integrating both sides of the resulting equation, by Lemma 2.7 and Theorem 2.8, we have

$$\frac{1}{2} \{\|f(t)\|_{2, \vartheta}^2 - \|f(s)\|_{2, \vartheta}^2\} + \int_s^t \left(\frac{1}{2} \|f(\tau)\|_{\sigma, \vartheta}^2 - C_{\vartheta} \|f(\tau)\|_\sigma^2 d\tau \right) \leq C_{\vartheta} \int_s^t \|g(\tau)\|_\infty \|f(\tau)\|_{\sigma, \vartheta}^2 d\tau. \quad (4.36)$$

Multiply (4.36) by $\frac{\delta_{0,2\vartheta-1}}{2C_\vartheta}$ and add it to (4.35). Then we have

$$\begin{aligned} & \sum_{0 \leq \bar{\vartheta} \leq 2\vartheta-1} \left(\frac{C_{\bar{\vartheta}}^*}{2} \{ \|f(t)\|_{2,\bar{\vartheta}/2}^2 - \|f(s)\|_{2,\bar{\vartheta}/2}^2 \} + \delta_{\bar{\vartheta},2\vartheta-1} \int_s^t \|f(\tau)\|_{\sigma,\bar{\vartheta}/2}^2 d\tau \right) - \delta' \{ \eta(t) - \eta(s) \} \\ & + \frac{\delta_{0,2\vartheta-1}}{2C_\vartheta} \left[\frac{1}{2} \{ \|f(t)\|_{2,\vartheta}^2 - \|f(s)\|_{2,\vartheta}^2 \} + \int_s^t \left(\frac{1}{2} \|f(\tau)\|_{\sigma,\vartheta}^2 - C_\vartheta \|f(\tau)\|_\sigma^2 \right) d\tau \right] \\ & \leq C_{\vartheta-1/2} \int_s^t \|g(\tau)\|_\infty \|f(\tau)\|_{\sigma,\vartheta-1/2}^2 d\tau + \frac{\delta_{0,2\vartheta-1}}{2} \int_s^t \|g(\tau)\|_\infty \|f(\tau)\|_{\sigma,\vartheta}^2 d\tau. \end{aligned}$$

Note that $\|\cdot\|_{2,\vartheta-1/2} \leq \|\cdot\|_{2,\vartheta}$, $\|\cdot\|_{\sigma,\vartheta-1/2} \leq \|\cdot\|_{\sigma,\vartheta}$. Choosing sequences of $C_{2\vartheta}^*$, $\delta_{\bar{\vartheta},2\vartheta}$, and C_ϑ such that

$$C_0^* = 1, \quad \delta_{0,0} = \delta', \quad C_0 = C,$$

$$C_\vartheta^* = \frac{\delta_{0,2\vartheta-1}}{2C_\vartheta}, \quad (4.37)$$

$$\delta_{\bar{\vartheta},2\vartheta} = \begin{cases} \frac{\delta_{0,2\vartheta-1}}{2}, & \text{if } \bar{\vartheta} = 0 \\ \delta_{\bar{\vartheta},2\vartheta-1}, & \text{if } \bar{\vartheta} = 1, \dots, \vartheta-1, \\ \frac{\delta_{0,2\vartheta-1}}{4C_\vartheta}, & \text{if } \bar{\vartheta} = 2\vartheta, \end{cases} \quad (4.38)$$

and

$$C_\vartheta = C_{\vartheta-1/2} + \frac{\delta_{0,2\vartheta-1}}{2}, \quad (4.39)$$

we have (4.35) for all ϑ .

Note that from (4.37) - (4.39), we have

$$\delta_{0,k} = \frac{\delta'}{2^k}, \quad \text{for } k = 1, 2, \dots, 2\vartheta.$$

$$C < C_\vartheta = C + \sum_{0 \leq \bar{\vartheta} \leq 2\vartheta-1} \frac{\delta_{0,\bar{\vartheta}}}{2} < C + \delta' < C + 1,$$

$$\frac{\delta'}{2^{2\vartheta}(C+1)} < C_\vartheta^* < \frac{\delta'}{2^{2\vartheta}C},$$

and

$$\delta_{2\vartheta,2\vartheta} = \frac{\delta_{0,2\vartheta-1}}{4C_\vartheta} = \frac{C_\vartheta^*}{2}.$$

Let $\varepsilon = \frac{\delta_{2\vartheta,2\vartheta}}{2C_\vartheta}$. By Remark 4.5, we can choose δ' small enough such that

$$\delta' \eta(t) \leq \frac{C_0^*}{4} \|f(t)\|_2^2 = \frac{1}{4} \|f(t)\|_2^2. \quad (4.40)$$

From (4.35), we have

$$\begin{aligned}
\frac{C_{\vartheta}^*}{2} \|f(t)\|_{2,\vartheta}^2 + \frac{\delta_{2\vartheta,2\vartheta}}{2} \int_s^t \|f(\tau)\|_{\sigma,\vartheta}^2 d\tau &\leq \frac{1}{4} \|f(s)\|_2^2 + \sum_{1 \leq \bar{\vartheta} \leq 2\vartheta} \frac{C_{\bar{\vartheta}}^*}{2} \|f(s)\|_{2,\bar{\vartheta}/2}^2 \\
&\leq \left(\frac{1}{4} + \frac{\delta'}{C} \right) \|f(s)\|_{2,\vartheta}^2 \\
&\leq \frac{1}{2} \|f(s)\|_{2,\vartheta}^2.
\end{aligned} \tag{4.41}$$

Taking $s = 0$ and dividing by $\frac{\delta_{2\vartheta,2\vartheta}}{2}$ both sides of (4.41), we have

$$2\|f(t)\|_{2,\vartheta}^2 + \int_0^t \|f(\tau)\|_{\sigma,\vartheta}^2 d\tau \leq \frac{2}{C_{\vartheta}^*} \|f(0)\|_{2,\vartheta}^2 \leq C 2^{2\vartheta} \|f(0)\|_{2,\vartheta}^2.$$

Therefore, we have (1.37).

Fix $\vartheta, k \geq 0$, by the Hölder inequality and (1.37),

$$\begin{aligned}
\|f\|_{2,\vartheta}^2 &= \int w^{2\vartheta} f^2 \\
&= \int \left(w^{2(\vartheta-\frac{1}{2})} f^2 \right)^{\frac{k}{k+1}} \left(w^{2(\vartheta+\frac{k}{2})} f^2 \right)^{\frac{1}{k+1}} \\
&\leq \left(\int w^{2(\vartheta-\frac{1}{2})} f^2 \right)^{\frac{k}{k+1}} \left(\int w^{2(\vartheta+\frac{k}{2})} f^2 \right)^{\frac{1}{k+1}} \\
&\leq \|f\|_{2,\vartheta-1/2}^{2\frac{k}{k+1}} \left(C 2^{2\vartheta+k} \mathcal{E}_{\vartheta+k/2}(0) \right)^{\frac{1}{k+1}}.
\end{aligned} \tag{4.42}$$

By Lemma 2.5,

$$\|f\|_{\sigma,\vartheta} \geq \|(1 + |v|)^{-1/2} f\|_{2,\vartheta} = \|f\|_{2,\vartheta-1/2}. \tag{4.43}$$

Combining (4.35), (4.42), and (4.43), we have

$$\begin{aligned}
&\sum_{1 \leq \bar{\vartheta} \leq 2\vartheta} \frac{C_{\bar{\vartheta}}^*}{2} \left(\|f(t)\|_{2,\bar{\vartheta}/2}^2 - \|f(s)\|_{2,\bar{\vartheta}/2}^2 \right) + \left\{ \frac{1}{2} \|f(t)\|_2^2 - \delta' \eta(t) \right\} - \left\{ \frac{1}{2} \|f(s)\|_2^2 - \delta' \eta(s) \right\} \\
&\leq -\frac{\delta_{2\vartheta,2\vartheta}}{2} \int_s^t \|f(\tau)\|_{\sigma,\vartheta}^2 d\tau \\
&\leq -\frac{\delta_{2\vartheta,2\vartheta}}{2} \int_s^t \|f(\tau)\|_{2,\vartheta-1/2}^2 d\tau \\
&\leq -\frac{\delta_{2\vartheta,2\vartheta}}{2} \int_s^t \left(C 2^{2\vartheta+k} \mathcal{E}_{\vartheta+k/2}(0) \right)^{-\frac{1}{k}} \|f(\tau)\|_{2,\vartheta}^{2\frac{k+1}{k}} d\tau.
\end{aligned} \tag{4.44}$$

Let

$$y(t) := \left\{ \frac{1}{2} \|f(t)\|_2^2 - \delta' \eta(t) \right\} + \sum_{1 \leq \bar{\vartheta} \leq 2\vartheta} \frac{C_{\bar{\vartheta}}^*}{2} \|f(t)\|_{2,\bar{\vartheta}/2}^2.$$

Then

$$\frac{C_{\vartheta}^*}{2} \|f(t)\|_{2,\vartheta}^2 \leq y(t) \leq \left(\frac{1}{2} + \sum_{\vartheta=1}^{2\vartheta} \frac{C_{\vartheta}^*}{2} \right) \|f(t)\|_{2,\vartheta}^2 \leq \|f(t)\|_{2,\vartheta}^2. \quad (4.45)$$

Combining (4.44), (4.45), we have

$$y(t) - y(s) \leq -\frac{\delta_{2\vartheta,2\vartheta}}{2} \int_s^t \left(C 2^{2\vartheta+k} \mathcal{E}_{\vartheta+k/2}(0) \right)^{-\frac{1}{k}} (y(\tau))^{\frac{k+1}{k}} d\tau.$$

Therefore, we have

$$y'(t) \leq -\frac{1}{2} \delta_{2\vartheta,2\vartheta} \left(C 2^{2\vartheta+k} \mathcal{E}_{\vartheta+k/2}(0) \right)^{-\frac{1}{k}} y(t)^{\frac{k+1}{k}} \leq -\frac{1}{2^{2\vartheta} C} \left(C 2^{2\vartheta+k} \mathcal{E}_{\vartheta+k/2}(0) \right)^{-\frac{1}{k}} y(t)^{\frac{k+1}{k}}. \quad (4.46)$$

Multiplying (4.46) by $-\frac{1}{k} y^{-\frac{k+1}{k}}$, we have

$$\partial_t \left(y(t)^{-\frac{1}{k}} \right) \geq \frac{1}{2^{2\vartheta} C k} \left(C 2^{2\vartheta+k} \mathcal{E}_{\vartheta+k/2}(0) \right)^{-\frac{1}{k}}.$$

Integrating above over $[0, t]$ yields

$$\begin{aligned} y(t)^{-\frac{1}{k}} &\geq \frac{t}{2^{2\vartheta} C k} \left(C 2^{2\vartheta+k} \mathcal{E}_{\vartheta+k/2}(0) \right)^{-\frac{1}{k}} + y(0)^{-\frac{1}{k}} \\ &\geq \frac{t}{2^{2\vartheta} C k} \left(C 2^{2\vartheta+k} \mathcal{E}_{\vartheta+k/2}(0) \right)^{-\frac{1}{k}} + (\|f(0)\|_{2,\vartheta}^2)^{-\frac{1}{k}} \\ &\geq \frac{(C 2^{2\vartheta+k} \mathcal{E}_{\vartheta+k/2}(0))^{-\frac{1}{k}}}{2^{2\vartheta} C} \left(\frac{t}{k} + 1 \right). \end{aligned}$$

Therefore,

$$\|f(t)\|_{2,\vartheta}^2 \leq \frac{2}{C_{\vartheta}^*} y(t) \leq C_{\vartheta,k} \mathcal{E}_{\vartheta+k/2}(0) \left(1 + \frac{t}{k} \right)^{-k},$$

where we use (4.45) in the first inequality. Thus we complete the proof. \square

Theorem 4.6. Assume (2.4). Let $\vartheta \in 2^{-1}\mathbb{N} \cup \{0\}$ and f be a classical solution of (1.7), (4.3). Then there exist $C, \varepsilon(\vartheta) > 0$ such that if $\|g\|_\infty < \varepsilon$, then

$$\sup_{0 \leq s < \infty} \mathcal{E}_{\vartheta}(f(s)) \leq C 2^{2\vartheta} \mathcal{E}_{\vartheta}(0), \quad (4.47)$$

and for any $t > 0$, $k \in \mathbb{N}$,

$$\|f(t)\|_{2,\vartheta} \leq C_{\vartheta,k} \mathcal{E}_{\vartheta+k/2}(0) \left(1 + \frac{t}{k} \right)^{-k/2}. \quad (4.48)$$

Sketch of proof. The proof can be done by choosing $\Gamma = 0$ in Theorem 1.4. \square

5. $L^2 - L^\infty$ ESTIMATE

5.1. **Local $L^2 - L^\infty$ estimate.** In this subsection we will derive a local L^∞ estimate for h .

$$\mathcal{M}_g^\vartheta h := (\partial_t + v \cdot \nabla_x - \bar{A}_g^\vartheta)h, \quad (5.1)$$

where \bar{A}_g^ϑ is defined as in (1.27).

Here we will refine the results about the $L^2 - L^\infty$ estimate in [19]. Comparing with [19], we have an additional term; $a_g \cdot \nabla_v f - 2 \frac{\partial_i w^\vartheta}{w^\vartheta} \sigma_G^{ij} \partial_j f$ and a diffusion matrix of \mathcal{M}_g^ϑ is not uniformly elliptic. Moreover, to get a $L^2 - L^\infty$ estimate for $\mathbb{T}^3 \times \mathbb{R}^3$, we need to know the local $L^2 - L^\infty$ estimate more explicitly.

Define $Q_n := [-t_n, 0] \times \mathbb{T}^3 \times B(0; R_n)$, for $t_n \geq t_{n+1}$ and $R_n \geq R_{n+1}$. The following estimates are refinements of Lemma 4 - 6 and Theorem 2 and Theorem 7 in [19].

Lemma 5.1 (Lemma 4 in [19]). *Assume (2.4). Let h be a nonnegative periodic function in x satisfying $\mathcal{M}_g^\vartheta h \leq 0$. Then h satisfies*

$$\int_{Q_1} |\nabla_v h|^2 \leq C \int_{Q_0} h^2 \quad (5.2)$$

$$\|h\|_{L_t^2 L_x^2 L_v^q(Q_1)}^2 \leq C \int_{Q_0} h^2 \quad (5.3)$$

$$\|h\|_{L_t^\infty L_x^2 L_v^2(Q_1)}^2 \leq C \int_{Q_0} h^2. \quad (5.4)$$

for some $q > 2$ and $C = \bar{C}(R_0) \left(1 + \frac{1}{t_0 - t_1} + \frac{1}{R_0 - R_1} + \frac{1}{(R_0 - R_1)^2}\right)$.

Proof. Consider a test function $\Phi \in C^\infty(\mathbb{R} \times \mathbb{T}^3 \times \mathbb{R}^3)$, periodic with respect to x and $\Phi(t, x, v) = 0$ for $|v| > R_0$. Multiplying (5.1) by $2h\Phi^2$ and integrating the resulting equation over $\mathcal{R} := [-t_0, s] \times \mathbb{T}^3 \times B(0; R_0)$ for some $s \in [-t_1, 0]$, then

$$\begin{aligned} & \int_{\mathcal{R}} \partial_t(h^2)\Phi^2 + \int_{\mathcal{R}} v \cdot \nabla_x(h^2)\Phi^2 \\ & \leq 2 \int_{\mathcal{R}} \nabla_v \cdot (\sigma_G \nabla_v h) h \Phi^2 + \int_{\mathcal{R}} a_g \cdot \nabla_v(h^2)\Phi^2 - 2 \int_{\mathcal{R}} \frac{\nabla_v(w^\vartheta)}{w^\vartheta} \cdot \sigma_G \nabla_v(h^2)\Phi^2, \end{aligned}$$

where σ_G is defined as in (1.12) with $G = \mu + \mu^{1/2}g$. Using the integration by parts and the positivity of σ_G , we have

$$\begin{aligned} & \int_{\mathcal{R}} \partial_t(h^2\Phi^2) + 2 \int_{\mathcal{R}} (\nabla_v h \cdot \sigma_G \nabla_v h) \Phi^2 \\ & \leq \int_{\mathcal{R}} h^2 \left(\partial_t(\Phi^2) + v \cdot \nabla_x(\Phi^2) - \nabla_v \cdot (\Phi^2 a_g) + 2 \nabla_v \cdot \left(\Phi^2 \sigma_G \frac{\nabla_v(w^\vartheta)}{w^\vartheta} \right) \right) - 4 \int_{\mathcal{R}} h \Phi \nabla_v \Phi \cdot \sigma_G \nabla_v h \\ & \leq \int_{\mathcal{R}} h^2 \left(\partial_t(\Phi^2) + v \cdot \nabla_x(\Phi^2) - \nabla_v \cdot (\Phi^2 a_g) + 2 \nabla_v \cdot \left(\Phi^2 \sigma_G \frac{\nabla_v(w^\vartheta)}{w^\vartheta} \right) \right) \\ & \quad + \int_{\mathcal{R}} (\nabla_v h \cdot \sigma_G \nabla_v h) \Phi^2 + C \int_{\mathcal{R}} (\nabla_v \Phi \cdot \sigma_G \nabla_v \Phi) h^2. \end{aligned}$$

Thus we have

$$\begin{aligned} & \int_{\mathcal{R}} \partial_t(h^2 \Phi^2) + \min(1, (1+R_0)^{-3}) \int_{\mathcal{R}} |\nabla_v h|^2 \Phi^2 \\ & \leq \bar{C}(d) \max(1, (1+R_0)^{-1}) \left(\|\partial_t \Phi\|_\infty \|\Phi\|_\infty + R_0 \|\nabla_x \Phi\|_\infty \|\Phi\|_\infty \right. \\ & \quad + \|\Phi\|_\infty \|a_g\|_\infty \|\nabla_v \Phi\|_\infty + \|\Phi\|_\infty^2 \|\nabla_v \cdot a_g\|_\infty \\ & \quad \left. + \|\nabla_v \Phi\|_\infty^2 + \|\Phi\|_\infty \|\nabla_v \Phi\|_\infty \left\| \sigma \frac{\nabla_v(w^\vartheta)}{w^\vartheta} \right\|_\infty + \|\Phi\|_\infty^2 \left\| \nabla_v \cdot \left(\sigma \frac{\nabla_v(w^\vartheta)}{w^\vartheta} \right) \right\|_\infty \right) \int_{\mathcal{R}} h^2. \end{aligned}$$

Choosing Φ such that $\Phi(-t_0, x, v) = 0$ and $\Phi = 1$ in Q_1 , we have

$$\int_{\mathbb{T}^3 \times B(0; R_1)} h^2(s) dx dv + \int_{\mathcal{R}} |\nabla_v h|^2 \leq \bar{C}(R_0) \left(1 + \frac{1}{t_0 - t_1} + \frac{1}{R_0 - R_1} + \frac{1}{(R_0 - R_1)^2} \right) \int_{\mathcal{R}} h^2. \quad (5.5)$$

Especially,

$$\sup_{s \in [-t_1, 0]} \int_{\mathbb{T}^3 \times B(0; R_1)} h^2(s) dx dv \leq \bar{C}(R_0) \left(1 + \frac{1}{t_0 - t_1} + \frac{1}{R_0 - R_1} + \frac{1}{(R_0 - R_1)^2} \right) \int_{Q_0} h^2.$$

Therefore, we prove (5.4). Choosing $s = 0$ in (5.5), we have

$$\int_{Q_1} |\nabla_v h|^2 \leq \bar{C}(R_0) \left(1 + \frac{1}{t_0 - t_1} + \frac{1}{R_0 - R_1} + \frac{1}{(R_0 - R_1)^2} \right) \int_{Q_0} h^2,$$

so we obtain (5.2). Moreover, the Sobolev inequality implies (5.3) □

Lemma 5.2 (Lemma 5 in [19]). *Assume (2.4). If h is a weak solution of (1.28), then*

$$\begin{aligned} \|D_x^{1/3} h\|_{L^2(Q_1)}^2 & \leq C \|h\|_{L^2(Q_0)}^2 \\ \|D_t^{1/3} h\|_{L^2(Q_1)}^2 & \leq C \|h\|_{L^2(Q_0)}^2 \end{aligned} \quad (5.6)$$

for some $C = \bar{C}(R_0) \left(1 + \frac{1}{t_0 - t_1} + \frac{1}{R_0 - R_1} + \frac{1}{(R_0 - R_1)^2} \right)$.

Proof. Let $R_{\frac{1}{2}} = \frac{R_1 + R_2}{2}$ and $Q_{\frac{1}{2}} = Q_{R_{\frac{1}{2}}}$. Define truncation functions χ_1 and $\chi_{1/2}$ such that

$$\begin{aligned} \chi_1 &= \begin{cases} 1, & \text{if } (t, x, v) \in Q_1 \\ 0, & \text{if } (t, x, v) \in Q_{\frac{1}{2}}^c, \end{cases} \\ \chi_{\frac{1}{2}} &= \begin{cases} 1, & \text{if } (t, x, v) \in Q_{\frac{1}{2}} \\ 0, & \text{if } (t, x, v) \in Q_0^c. \end{cases} \end{aligned}$$

Let $h_i = h \chi_i$, for $i = 1, \frac{1}{2}$. Then we have

$$\begin{aligned} (\partial_t + v \cdot \nabla_x) h_1 &= \nabla_v \cdot H_1 + H_0 \quad \text{in } (-\infty, 0] \times \mathbb{R}^6, \\ H_1 &= \chi_1 \sigma_G \nabla_v h_{\frac{1}{2}}, \\ H_0 &= -\nabla_v \chi_1 \cdot \sigma_G \nabla_v h_{\frac{1}{2}} + \alpha_1 h_{\frac{1}{2}} + \chi_1 a_g \cdot \nabla_v h_{1/2} - 2\chi_1 \frac{\nabla_v(w^\vartheta)}{w^\vartheta} \cdot \sigma_G \nabla_v h_{1/2}, \\ \alpha_1 &= (\partial_t + v \cdot \nabla_x) \chi_1, \end{aligned}$$

where σ_G is defined as in (1.12) with $G = \mu + \mu^{1/2}g$. By Lemma 5.1,

$$\|H_0\|_{L^2(\mathbb{R}^7)} + \|H_1\|_{L^2(\mathbb{R}^7)} \leq C\|h\|_{L^2(Q_0)}$$

with C as in the statement. Applying Theorem 1.3 in [4] with $p = 2$, $r = 0$, $\beta = 1$, $m = 1$, $\kappa = 1$ and $\Omega = 1$ yields (5.6). \square

Lemma 5.3 (Lemma 6 in [19]). *Under the assumptions of Lemma 5.1, there exists $p > 2$ such that*

$$\|h\|_{L_t^2 L_x^p L_v^2(Q_1)}^2 \leq C\|h\|_{L^2(Q_0)}^2 \quad (5.7)$$

with the same C as in Lemma 5.1.

Proof. The proof is exactly the same as in the proof of Lemma 6 in [19]. We omit the proof. \square

The following Lemma is a consequence of Lemma 5.1, 5.2. We omit the proof.

Lemma 5.4. *Under the assumptions of Lemma 5.2, we have*

$$\|h\|_{H_{x,v,t}^s(Q_1)} \leq C\|h\|_{L^2(Q_0)}$$

with the same C as in Lemma 5.1 and $s = 1/3$.

Lemma 5.5 (Theorem 2 in [19]). *Under the assumptions of Lemma 5.1, there exists $q > 2$ such that*

$$\|h\|_{L^q(Q_1)}^2 \leq C\|h\|_{L^2(Q_0)}^2 \quad (5.8)$$

with the same C as in Lemma 5.1.

Proof. The proof is exactly the same as in the proof of Theorem 2 in [19]. We omit the proof. \square

Lemma 5.6 (Theorem 7 in [19]). *Assume (2.4). Let h be a nonnegative periodic function in x satisfying $\mathcal{M}_g^\vartheta h \leq 0$. Then, there exists $m > 1$ such that*

$$\|h\|_{L^\infty(Q_\infty)} \leq \bar{C}(R_0)^m \left(1 + \frac{1}{\min(t_0 - t_\infty, (R_0 - R_\infty)^2)}\right)^m \|h\|_{L^2(Q_0)},$$

where $Q_0 = [-t_0, 0] \times \mathbb{T}^3 \times [-R_0, R_0]$ and $Q_\infty = [-t_\infty, 0] \times \mathbb{T}^3 \times [-R_\infty, R_\infty]$.

Proof. Let $\kappa := q/2 > 1$. Since $|h|^{q_n}$, $q_n > 1$, is also a sub-solution of (1.28), by Lemma 5.5

$$\||h|^{q_n}\|_{L^q(Q_{n+1})}^2 \leq C_n \||h|^{q_n}\|_{L^2(Q_n)}^2,$$

where $C_n = \bar{C}(R_0) \left(1 + \frac{1}{t_n - t_{n+1}} + \frac{1}{R_n - R_{n+1}} + \frac{1}{(R_n - R_{n+1})^2}\right)$. Changing $\|\cdot\|_q$ to $\|\cdot\|_2$ yields

$$\||h|^{\kappa q_n}\|_{L^2(Q_{n+1})}^2 \leq C_n^\kappa \||h|^{q_n}\|_{L^2(Q_n)}^{2\kappa}.$$

Let $q_n := \kappa^n$, then after iteration we have

$$\||h|^{q_n}\|_{L^2(Q_n)}^2 \leq \prod_{j=1}^n C_{n-j}^{\kappa^j} \|h\|_{L^2(Q_0)}^{2\kappa^n}.$$

Changing $\|\cdot\|_2$ to $\|\cdot\|_{2q_n}$, we have

$$\|h\|_{L^{2q_n}(Q_n)}^2 \leq \prod_{j=1}^n C_{n-j}^{\kappa^j - n} \|h\|_{L^2(Q_0)}^2 = \prod_{j=0}^{n-1} C_j^{\kappa^{-j}} \|h\|_{L^2(Q_0)}^2.$$

Choosing $t_n - t_{n+1} = \alpha(t_0 - t_\infty)n^{-4}$ and $R_n - R_{n+1} = \beta(R_0 - R_\infty)n^{-2}$, we have

$$C_j^{\kappa^{-j}} \leq \bar{C}^{\kappa^{-j}} \left(C' \left(1 + \frac{1}{\min(t_0 - t_\infty, (R_0 - R_\infty)^2)} \right) \right)^{j^4 \kappa^{-j}}.$$

Thus,

$$\prod_{j=0}^{\infty} C_j^{\kappa^{-j}} \leq C'^m \bar{C}(R_0)^m \left(1 + \frac{1}{\min(t_0 - t_\infty, (R_0 - R_\infty)^2)} \right)^m$$

for some $m > 1$. So the proof is complete. \square

Lemma 5.7. *Assume (2.4). If $h_+ = \max\{h, 0\}$, where h is a subsolution of (1.28), then h_+ is a subsolution.*

Proof. Approximate a convex function $Q(h) \rightarrow h_+$ and then use the convexity of $Q(h)$ such that $Q'(h) > 0$ and $Q''(h) > 0$. Applying $Q(h)$ to the equation (1.28), we complete the proof. \square

Let h be a weak solution. Then since $|h| = h_+ - h_-$ and $h_+ = \max\{h, 0\}$ are subsolutions (maximum of two subsolutions is a subsolution) and $h_- = \min\{-h, 0\}$ is a supersolution (minimum of two supersolutions is a supersolution), we can apply Lemma above to both h_+ and $-h_-$. Thus we obtain:

Lemma 5.8 (Theorem 7 in [19]). *Assume (2.4). Let h be a sub-solution of (1.28). Then, there exists $m > 1$ such that*

$$\|h\|_{L^\infty(Q_\infty)} \leq \bar{C}(R_0)^m \left(1 + \frac{1}{\min(t_0 - t_\infty, (R_0 - R_\infty)^2)} \right)^m \|h\|_{L^2(Q_0)},$$

where $Q_0 = [-t_0, 0] \times \mathbb{T}^3 \times [-R_0, R_0]$ and $Q_\infty = [-t_\infty, 0] \times \mathbb{T}^3 \times [-R_\infty, R_\infty]$.

5.2. $L^2 - L^\infty$ estimate for (1.25). We now consider (1.25) and let f be a solution of (1.25). Then we split f into two parts:

$$f = f \mathbf{1}_{\{|v| \leq M\}} + f \mathbf{1}_{\{|v| \geq M\}} =: f_1 + f_2.$$

Let $U(t, s)h$ be a solution of (1.28) corresponding to the initial times s with the initial data h . Then

$$\begin{aligned} f_1(t, x, v) &= \mathbf{1}_{\{|v| \leq M\}} U(t, 0) f_0 + \mathbf{1}_{\{|v| \leq M\}} \int_0^t U(t, \tau) \bar{K}_g^\vartheta f(\tau) d\tau \\ &= \mathbf{1}_{\{|v| \leq M\}} U(t, 0) f_0 + \int_0^t \mathbf{1}_{\{|v| \leq M\}} U(t, \tau) \bar{K}_g^\vartheta f(\tau) d\tau. \end{aligned}$$

Then we first obtain the L^∞ estimates for f_1 :

Lemma 5.9. *Assume (2.4). Let f be a weak solution of (1.6), (1.7), and (1.25) in a periodic box in the sense of Definition 4.1, then there exist $C, \beta > 0$ satisfying the following property: for any $Z, s, k > 1$, and $\vartheta, l \in \mathbb{N} \cup \{0\}$, there exists $C_{\vartheta, l}$ such that*

$$\begin{aligned} &\left\| \mathbf{1}_{|v| < Zs^k} f^\vartheta(s) \right\|_\infty \\ &\leq C_{\vartheta, l} \left(Zs^k \right)^\beta (1+s)^{-l} \|f_0\|_{2, \vartheta+l} + \frac{C}{1+Zs^k} \sup_{s' \in (s-1, s)} \|f^\vartheta(s')\|_\infty. \end{aligned} \tag{5.9}$$

Proof. By the Duhamel principle,

$$\begin{aligned} \|1_{|v| < Zs^k} f^\vartheta(s)\|_{L^\infty} &\leq \|1_{|v| < Zs^k} U(s, s-1) f^\vartheta(s-1)\|_{L^\infty} \\ &\quad + \int_0^{1-\varepsilon} \|1_{|v| \leq Zs^k} U(s, s-1+\tau) \bar{K}_g^\vartheta f(s-1+\tau)\|_\infty d\tau \\ &\quad + \int_{1-\varepsilon}^1 \|1_{|v| \leq Zs^k} U(s, s-1+\tau) \bar{K}_g^\vartheta f(s-1+\tau)\|_\infty d\tau \\ &= (i) + (ii) + (iii), \end{aligned}$$

where \bar{K}_g^ϑ is defined as in (1.26) and ε is a constant which will be chosen later. By Lemma 5.8, there exists $m > 0$ such that

$$(i) \leq C \left(Zs^k \right)^m \left(\int_0^1 \|U(s', s-1) f^\vartheta(s-1)\|_2^2 ds' \right)^{1/2}.$$

By Theorem 1.4 and Lemma 3.3, for every integer l , there exists C_l such that

$$\begin{aligned} \|U(s', s-1) f^\vartheta(s-1)\|_2 &\leq C \|f^\vartheta(s-1)\|_2 = C \|f(s-1)\|_{2,\vartheta} \\ &\leq C_{\vartheta,l} \left(1 + \frac{s-1}{l} \right)^{-l} \|f_0\|_{2,\vartheta+l} \\ &\leq C_{\vartheta,l} (1+s)^{-l} \|f_0\|_{2,\vartheta+l}. \end{aligned}$$

Thus

$$(i) \leq C_{\vartheta,l} \left(Zs^k \right)^m (1+s)^{-l} \|f_0\|_{2,\vartheta+l}.$$

By the maximum principle and (2.15),

$$(iii) \leq C\varepsilon \sup_{s' \in (s-1, s)} \|f^\vartheta(s')\|_\infty.$$

By Lemma 5.8,

$$\begin{aligned} &\|1_{|v| \leq Zs^k} U(s, s-1+\tau) \bar{K}_g^\vartheta f(s-1+\tau)\|_\infty \\ &\leq C \left(Zs^k \right)^m \left(1 + \frac{1}{1-\tau} \right)^m \left(\int_{s-1+\tau}^s \|1_{|v| < 2Zs^k} U(s', s-1+\tau) \bar{K}_g^\vartheta f(s-1+\tau)\|_2^2 ds' \right)^{1/2}. \end{aligned}$$

By (2.17) and Theorem 1.4, for any $N > 0$,

$$\begin{aligned} (ii) &\leq C \int_0^{1-\varepsilon} \left(Zs^k \right)^m \left(1 + \frac{1}{1-\tau} \right)^m \left(\int_{s-1+\tau}^s \|\bar{K}_f^\vartheta f(s-1+\tau)\|_2^2 ds' \right)^{1/2} d\tau \\ &\leq C \left(Zs^k \right)^m \left(1 + \frac{1}{\varepsilon} \right)^m \int_0^1 \left(N^2 \|f^\vartheta(s-1+\tau)\|_2 + \frac{1}{N} \|f^\vartheta(s-1+\tau)\|_\infty \right) d\tau \\ &\leq \left(Zs^k \right)^m \left(1 + \frac{1}{\varepsilon} \right)^m \left(C_{\vartheta,l} N^2 (1+s)^{-l} \|f_0\|_{2,\vartheta+l} + \frac{C}{N} \sup_{s' \in (s-1, s)} \|f^\vartheta(s')\|_\infty \right). \end{aligned}$$

Choose $\varepsilon^{-1} = 1 + Zs^k$ and $N = (1 + Zs^k)^{2m+1}$. Then

$$(i) + (ii) + (iii) \leq C_{\vartheta,l} \left(Zs^k \right)^\beta (1+s)^{-l} \|f_0\|_{2,\vartheta+l} + \frac{C}{1 + Zs^k} \sup_{s' \in (s-1, s)} \|f^\vartheta(s')\|_\infty,$$

where $\beta = 6m + 2 > 0$. □

Based on the above results, we will prove Theorem 1.5.

Proof of Theorem 1.5. Choose ε as in Lemma 2.4. By Lemma 5.9, there exists l such that

$$\left\| 1_{|v| < Zs^k} f^\vartheta(s) \right\|_{L^\infty} \leq C_{\vartheta, l, Z} (1+s)^{-2} \|f_0\|_{2, \vartheta+l} + \frac{C}{1+Zs^k} \sup_{s' \in (s-1, s)} \|f^\vartheta(s')\|_\infty.$$

Therefore by the Duhamel principle, the maximum principle, and (2.16),

$$\begin{aligned} \|f^\vartheta(n+1)\|_{L^\infty} &\leq \|U(n+1, n) f^\vartheta(n)\|_{L^\infty} \\ &\quad + \int_0^1 \left\| U(n+1, n+s) \bar{K}_g^\vartheta \left(1_{|v| < Z(n+s)^k} f(n+s) + 1_{|v| > Z(n+s)^k} f(n+s) \right) \right\|_{L^\infty} ds \\ &\leq \|f^\vartheta(n)\|_{L^\infty} + C_{\vartheta, l, Z} \int_0^1 (1+n+s)^{-2} \|f_0\|_{2, \vartheta+l} \\ &\quad + \frac{C}{1+Z(n+s)^k} \sup_{s' \in (n+s-1, n+s)} \|f^\vartheta(s')\|_\infty ds \\ &\quad + \int_0^1 (1+Z(n+s)^k)^{-1} \|f^\vartheta(n+s)\|_\infty ds \\ &\leq \|f^\vartheta(n)\|_{L^\infty} + C_{\vartheta, l, Z} (1+n)^{-2} \|f_0\|_{2, \vartheta+l} + C(Zn^k)^{-1} \sup_{s' \in [n-1, n+1]} \|f^\vartheta(s')\|_\infty, \end{aligned}$$

where \bar{K}_g^ϑ is defined as in (1.26). After iteration,

$$\begin{aligned} \|f^\vartheta(n+1)\|_{L^\infty} &\leq \|f^\vartheta(1)\|_{L^\infty} + C_{\vartheta, l, Z} \sum_{\bar{n}=1}^n (1+\bar{n})^{-2} \|f_0\|_{2, \vartheta+l} \\ &\quad + CZ^{-1} \sum_{\bar{n}=1}^n \bar{n}^{-k} \sup_{s \in [0, n+1]} \|f^\vartheta(s)\|_\infty. \end{aligned}$$

Choose large k and Z such that $-k < -1$ and $CZ^{-1} \sum_{\bar{n}=1}^\infty \bar{n}^{k(-1)} \leq \varepsilon_0$, where ε_0 will be determined later. Then

$$\|f^\vartheta(n+1)\|_{L^\infty} \leq \|f^\vartheta(1)\|_{L^\infty} + C_{\vartheta, l, Z} \|f_0\|_{2, \vartheta+l} + \varepsilon_0 \sup_{s \in [0, n+1]} \|f^\vartheta(s)\|_\infty.$$

Since n is an arbitrary integer,

$$\sup_{s=1, 2, \dots, n+1} \|f^\vartheta(s)\|_{L^\infty} \leq \|f^\vartheta(1)\|_{L^\infty} + C_{\vartheta, l, Z} \|f_0\|_{2, \vartheta+l} + \varepsilon_0 \sup_{s \in [0, n+1]} \|f^\vartheta(s)\|_\infty.$$

By the Duhamel principle, the maximum principle, and (2.16).

$$\begin{aligned} \|f^\vartheta(n+t)\|_{L^\infty} &\leq \|U(n+t, n) f^\vartheta(n)\|_{L^\infty} + \int_0^t \left\| U(n+t, n+s) \bar{K}_g^\vartheta f(n+s) \right\|_{L^\infty} ds \\ &\leq \|f^\vartheta(n)\|_{L^\infty} + \int_0^t \left\| \bar{K}_g^\vartheta f(n+s) \right\|_{L^\infty} ds \\ &\leq \|f^\vartheta(n)\|_{L^\infty} + C \int_0^t \left\| f^\vartheta(n+s) \right\|_{L^\infty} ds, \end{aligned}$$

for $t \in [0, 1]$. By the Gronwall inequality,

$$\|f^\vartheta(n+t)\|_{L^\infty} \leq C\|f^\vartheta(n)\|_\infty, \text{ for all } t \in [0, 1].$$

Therefore,

$$\sup_{s \in [0, n+1]} \|f^\vartheta(s)\|_{L^\infty} \leq C\|f_0^\vartheta\|_{L^\infty} + C_{\vartheta, l}\|f_0\|_{2, \vartheta+l} + C\varepsilon_0 \sup_{s \in [0, n+1]} \|f^\vartheta(s)\|_\infty.$$

Now, we choose a small ε_0 satisfying $C\varepsilon_0 < 1/2$, and then absorb the last term on the RHS to the LHS. Then, we have (1.39) in case of $\vartheta_0 = 0$ by taking $l_0(0) = l$.

By (5.9), there exist $C, l_1(\vartheta_0)$ such that

$$\|1_{|v| < (1+t)^{\vartheta_0}} f^\vartheta(t)\|_\infty \leq C(1+t)^{-\vartheta_0}(\|f_0\|_{2, \vartheta+l_1(\vartheta_0)} + \|f\|_{\infty, \vartheta}).$$

Thus, by Proposition 2.1, we have

$$\begin{aligned} \|f(t)\|_{\infty, \vartheta} &\leq \|1_{|v| < (1+t)} f(t)\|_{\infty, \vartheta} + \|1_{|v| \geq (1+t)} f(t)\|_{\infty, \vartheta} \\ &\leq C(1+t)^{-\vartheta_0} \left(\|f_0\|_{2, \vartheta+l_1(\vartheta_0)} + \sup_{0 \leq s \leq t} \|f(s)\|_{\infty, \vartheta} \right) + C(1+t)^{-\vartheta_0} \|1_{|v| \geq (1+t)} f(t)\|_{\infty, \vartheta+\vartheta_0} \\ &\leq C(1+t)^{-\vartheta_0} \left(\|f_0\|_{2, \vartheta+l_1(\vartheta_0)} + \sup_{0 \leq s \leq t} \|f(s)\|_{\infty, \vartheta+\vartheta_0} \right) \\ &\leq C(1+t)^{-\vartheta_0} (\|f_0\|_{2, \vartheta+l_1(\vartheta_0)} + \|f_0\|_{2, \vartheta+\vartheta_0+l_0(0)} + \|f_0\|_{\infty, \vartheta+\vartheta_0}) \\ &\leq C(1+t)^{-\vartheta_0} \|f_0\|_{\infty, \vartheta+l_0(\vartheta_0)}, \end{aligned}$$

where $l_0(\vartheta_0) = \max\{l_1(\vartheta_0), \vartheta_0 + l_0(0)\} + 2$. □

Lemma 5.10. *Assume (2.4). Let f be a strong solution of (1.6), (1.7), and (1.25) in a periodic box. Let $\beta > 0$ and $p > 2$ be given constants. Then there exist $l \in \mathbb{N}$ and $C_{\beta, l}$ such that*

$$\begin{aligned} \left(\int_0^t \|f(s)\|_{p, \beta}^p ds \right)^{1/p} &\leq C_{\beta, l_0} \|f_0\|_{2, \beta+l}^{2/p} (\|f_0\|_{\infty, \beta} + \|f_0\|_{2, \beta+l})^{(p-2)/p} \\ &\leq C_{\beta, l_0} (\|f_0\|_{\infty, \beta} + \|f_0\|_{2, \beta+l}). \end{aligned} \tag{5.10}$$

Proof. By Theorem 1.4 and Theorem 1.5, there exist $l \in \mathbb{N}$ and $C_{\beta, l}$ such that

$$\begin{aligned} \|f(s)\|_{2, \beta} &\leq C_{\beta, l_0} (1+s)^{-l} \|f_0\|_{2, \beta+l} \\ \|f(s)\|_{\infty, \beta} &\leq C_{\beta, l_0} (\|f_0\|_{\infty, \beta} + \|f_0\|_{2, \beta+l}). \end{aligned}$$

By the interpolation, we have

$$\|f(s)\|_{p, \beta}^p \leq (C_{\beta, l_0})^p (1+s)^{-2l} \|f_0\|_{2, \beta+l}^2 (\|f_0\|_{\infty, \beta} + \|f_0\|_{2, \beta+l})^{p-2}.$$

Taking the integral over $s \in (0, \infty)$, we have the first inequality of (5.10). The second inequality of (5.10) comes from the Young inequality, then we complete the proof. □

5.3. $L^2 - L^\infty$ estimate for (4.3). We will derive another type of $L^2 - L^\infty$ estimate to obtain a uniform Hölder estimate for a weak solution of (1.25) in the sense of Definition 4.1. The proof is similar to the case of Section 5.2.

Let us multiply (4.3) by w^ϑ , then $h := w^\vartheta f = f^\vartheta$ satisfies

$$(\partial_t + v \cdot \nabla_x - \bar{A}_g^\vartheta)h = \tilde{K}_g^\vartheta h, \quad (5.11)$$

where

$$\tilde{K}_g^\vartheta h = \left(2 \frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}} \sigma_G^{ij} - \frac{\partial_{ij} w^\vartheta}{w^\vartheta} \sigma_G^{ij} - \frac{\partial_j w^\vartheta}{w^\vartheta} \partial_i \sigma_G^{ij} - \frac{\partial_i w^\vartheta}{w^\vartheta} a_g^i \right) h. \quad (5.12)$$

Similar to Definition 4.1, we can define a weak solution of (5.11).

Then we split f^ϑ into two parts:

$$f^\vartheta = f^\vartheta \mathbf{1}_{\{|v| \leq M\}} + f^\vartheta \mathbf{1}_{\{|v| \geq M\}} =: f_1 + f_2.$$

Let $U^\vartheta(t, s)f_0$ be a weak solution of (5.1) in the sense of Definition 3.1 corresponding to the initial data f_0 with the initial time $t = s$, then we have

$$\begin{aligned} f_1(t) &= \mathbf{1}_{|v| \leq M} U^\vartheta(t, 0) f_0^\vartheta + \mathbf{1}_{|v| \leq M} \int_0^t U^\vartheta(t, \tau) \tilde{K}_g^\vartheta f^\vartheta(\tau) d\tau \\ &= \mathbf{1}_{|v| \leq M} U^\vartheta(t, 0) f_0^\vartheta + \int_0^t \mathbf{1}_{|v| \leq M} U^\vartheta(t, \tau) \tilde{K}_g^\vartheta f^\vartheta(\tau) d\tau. \end{aligned}$$

Lemma 5.11. *Assume (2.4). There exists $C = C_\vartheta > 0$ such that*

$$\|\tilde{K}_g^\vartheta f^\vartheta\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C \|f^\vartheta\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}, \quad (5.13)$$

$$\|\tilde{K}_g^\vartheta \mathbf{1}_{|v| > M} f^\vartheta\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C(1 + M)^{-1} \|f^\vartheta\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}, \quad (5.14)$$

and

$$\|\tilde{K}_g^\vartheta f^\vartheta\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C \|f^\vartheta\|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)}. \quad (5.15)$$

Proof. Since

$$\tilde{K}_g^\vartheta f^\vartheta = \left(2 \frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}} \sigma_G^{ij} - \frac{\partial_{ij} w^\vartheta}{w^\vartheta} \sigma_G^{ij} - \frac{\partial_j w^\vartheta}{w^\vartheta} \partial_i \sigma_G^{ij} - \frac{\partial_i w^\vartheta}{w^\vartheta} a_g^i \right) f^\vartheta$$

and by Lemma 2.2,

$$\left| 2 \frac{\partial_i w^\vartheta \partial_j w^\vartheta}{w^{2\vartheta}} \sigma_G^{ij} \right| + \left| \frac{\partial_{ij} w^\vartheta}{w^\vartheta} \sigma_G^{ij} \right| + \left| \frac{\partial_j w^\vartheta}{w^\vartheta} \partial_i \sigma_G^{ij} \right| + \left| \frac{\partial_i w^\vartheta}{w^\vartheta} a_g^i \right| \leq C(1 + |v|)^{-1}.$$

So the proof is complete. \square

Lemma 5.12. *Assume (2.4). Let f be a weak solution of (4.3) in a periodic box in the sense of Definition 3.1, then there exist $C, \beta > 0$ satisfying the following property: for any $Z, s > 1, \vartheta$, and $k > 0$, and $l \in \mathbb{N}$, there exists $C_{\vartheta, l}$ such that*

$$\begin{aligned} &\left\| \mathbf{1}_{|v| < Zs^k} f^\vartheta(s) \right\|_\infty \\ &\leq C_{\vartheta, l} \left(Zs^k \right)^\beta (1 + s)^{-l} \|f_0\|_{2, \vartheta + l} + \frac{C}{1 + Zs^k} \sup_{s' \in (s-1, s)} \|f^\vartheta(s')\|_\infty, \end{aligned} \quad (5.16)$$

for any $s \geq 1$.

Proof. By the Duhamel principle,

$$\begin{aligned} \|1_{|v| < Zs^k} f^\vartheta(s)\|_{L^\infty} &\leq \|1_{|v| < Zs^k} U^\vartheta(s, s-1) f^\vartheta(s-1)\|_{L^\infty} \\ &\quad + \int_0^{1-\varepsilon} \|1_{|v| \leq Zs^k} U^\vartheta(s, s-1+\tau) \tilde{K}_g^\vartheta f^\vartheta(s-1+\tau)\|_\infty d\tau \\ &\quad + \int_{1-\varepsilon}^1 \|1_{|v| \leq Zs^k} U^\vartheta(s, s-1+\tau) \tilde{K}_g^\vartheta f^\vartheta(s-1+\tau)\|_\infty d\tau \\ &= (i) + (ii) + (iii). \end{aligned}$$

By Lemma 5.8, there exists $m > 0$ such that

$$(i) \leq C \left(Zs^k \right)^m \left(\int_{s-1}^s \|U^\vartheta(s', s-1) f^\vartheta(s-1)\|_2^2 ds' \right)^{1/2}.$$

By Theorem 4.6 and Lemma 3.3, for every integer l , there exists C_l such that

$$\begin{aligned} \|U^\vartheta(s', s-1) f^\vartheta(s-1)\|_2 &\leq C \|f^\vartheta(s-1)\|_2 = C \|f(s-1)\|_{2,\vartheta} \\ &\leq C_{\vartheta,l} \left(1 + \frac{s-1}{l} \right)^{-l} \|f_0\|_{2,\vartheta+l} \\ &\leq C_{\vartheta,l} (1+s)^{-l} \|f_0\|_{2,\vartheta+l}. \end{aligned}$$

Thus

$$(i) \leq C_{\vartheta,l} \left(Zs^k \right)^m (1+s)^{-l} \|f_0\|_{2,\vartheta+l}.$$

By the maximum principle and (5.13),

$$(iii) \leq C\varepsilon \sup_{s' \in (s-1, s)} \|f^\vartheta(s')\|_\infty.$$

By Lemma 5.8,

$$\begin{aligned} &\|1_{|v| \leq Zs^k} U^\vartheta(s, s-1+\tau) \tilde{K}_g^\vartheta f(s-1+\tau)\|_\infty \\ &\leq C \left(Zs^k \right)^m \left(1 + \frac{1}{1-\tau} \right)^m \left(\int_s^{s-1+\tau} \|1_{|v| < 2Zs^k} U^\vartheta(s', s-1+\tau) \tilde{K}_g^\vartheta f(s-1+\tau)\|_2^2 ds' \right)^{1/2}. \end{aligned}$$

By (5.15) and Theorem 4.6, for any $N > 0$,

$$\begin{aligned} (ii) &\leq C \int_0^{1-\varepsilon} \left(Zs^k \right)^m \left(1 + \frac{1}{1-\tau} \right)^m \left(\int_0^{1-\tau} \|\tilde{K}_g^\vartheta f^\vartheta(s-1+\tau)\|_2^2 ds' \right)^{1/2} d\tau \\ &\leq C \left(Zs^k \right)^m \left(1 + \frac{1}{\varepsilon} \right)^m \int_0^1 C \|f^\vartheta(s-1+\tau)\|_2 d\tau \\ &\leq \left(Zs^k \right)^m \left(1 + \frac{1}{\varepsilon} \right)^m C_{\vartheta,l} (1+s)^{-l} \|f_0\|_{2,\vartheta-l}. \end{aligned}$$

Choose $\varepsilon^{-1} = 1 + Zs^k$. Then

$$(i) + (ii) + (iii) \leq C_{\vartheta,l} \left(Zs^k \right)^\beta (1+s)^{-l} \|f_0\|_{2,\vartheta-l} + \frac{C}{1+Zs^k} \sup_{s' \in (s-1, s)} \|f^\vartheta(s')\|_\infty,$$

where $\beta = 2m + 1$. □

Theorem 5.13. *Assume (2.4). Let f be a weak solution of (1.7), (4.3) in a periodic box in the sense of Definition 3.1. Then there exists l such that for every $\vartheta > 0$,*

$$\|f^\vartheta(t)\|_{L^\infty} \leq C\|f_0^\vartheta\|_{L^\infty} + C_{\vartheta,l}\|f_0\|_{2,\vartheta+l} \leq C\|f_0\|_{\infty,\vartheta+l_0}, \quad \text{for any } t > 0, \quad (5.17)$$

where $l_0 = l + 2$.

Proof. By Lemma 5.12, there exists l_0 such that for $l > l_0$,

$$\left\| 1_{|v| < Zs^k} f^\vartheta(s) \right\|_{L^\infty} \leq C_{\vartheta,l,Z}(1+s)^{-2}\|f_0\|_{2,\vartheta+l} + \frac{C}{1+Zs^k} \sup_{s' \in (s-1,s)} \|f^\vartheta(s')\|_\infty.$$

Therefore by the Duhamel principle, the maximum principle, and (5.14),

$$\begin{aligned} \|f^\vartheta(n+1)\|_{L^\infty} &\leq \|U^\vartheta(n+1,n)f^\vartheta(n)\|_{L^\infty} \\ &\quad + \int_n^{n+1} \left\| U^\vartheta(n,n-1+s)\tilde{K}_g^\vartheta \left(1_{|v| < Z(n+s)^k} f^\vartheta(n+s) + 1_{|v| > Z(n+s)^k} f^\vartheta(n+s) \right) \right\|_{L^\infty} ds \\ &\leq \|f^\vartheta(n)\|_{L^\infty} + C_{\vartheta,l,Z} \int_0^1 (1+n+s)^{-2}\|f_0\|_{2,\vartheta+l} \\ &\quad + \frac{C}{1+Z(n+s)^k} \sup_{s' \in (n+s-1,n+s)} \|f^\vartheta(s')\|_\infty ds \\ &\quad + \int_0^1 (1+Z(n+s)^k)^{-1} \|f^\vartheta(n+s)\|_\infty ds \\ &\leq \|f^\vartheta(n)\|_{L^\infty} + C_{\vartheta,l,Z}(1+n)^{-2}\|f_0\|_{2,\vartheta+l} \\ &\quad + C(Zn^k)^{-1} \sup_{s' \in [n-1,n+1]} \|f^\vartheta(s')\|_\infty. \end{aligned}$$

After iteration,

$$\begin{aligned} \|f^\vartheta(n+1)\|_{L^\infty} &\leq \|f^\vartheta(1)\|_{L^\infty} + C_{\vartheta,l,Z} \sum_{\bar{n}=1}^n (1+\bar{n})^{-2}\|f_0\|_{2,\vartheta+l} \\ &\quad + CZ^{-1} \sum_{\bar{n}=1}^n \bar{n}^{k(-1)} \sup_{s \in [0,n+1]} \|f^\vartheta(s)\|_\infty. \end{aligned}$$

Choose large k and Z such that $k(-1) < -1$ and $CZ^{-1} \sum_{\bar{n}=1}^\infty \bar{n}^{k(-1)} \leq \varepsilon$, where ε will be determined later. Then

$$\|f^\vartheta(n+1)\|_{L^\infty} \leq \|f^\vartheta(1)\|_{L^\infty} + C_{\vartheta,l,Z}\|f_0\|_{2,\vartheta+l} + \varepsilon \sup_{s \in [0,n+1]} \|f^\vartheta(s)\|_\infty.$$

Since n is an arbitrary integer,

$$\sup_{s=1,2,\dots,n+1} \|f^\vartheta(s)\|_{L^\infty} \leq \|f^\vartheta(1)\|_{L^\infty} + C_{\vartheta,l,Z}\|f_0\|_{2,\vartheta+l} + \varepsilon \sup_{s \in [0,n+1]} \|f^\vartheta(s)\|_\infty.$$

By the Duhamel principle, the maximum principle, and (5.14)

$$\begin{aligned} \|f^\vartheta(n+t)\|_{L^\infty} &\leq \|U^\vartheta(n+t, n)f^\vartheta(n)\|_{L^\infty} + \int_n^{n+t} \|U^\vartheta(n+t, n+s)\tilde{K}_g^\vartheta f^\vartheta(n+s)\|_{L^\infty} ds \\ &\leq \|f^\vartheta(n)\|_{L^\infty} + \int_0^t \|\tilde{K}_g^\vartheta f^\vartheta(n+s)\|_{L^\infty} ds \\ &\leq \|f^\vartheta(n)\|_{L^\infty} + C \int_0^t \|f^\vartheta(n+s)\|_{L^\infty} ds. \end{aligned}$$

By the Gronwall inequality,

$$\|f^\vartheta(n+t)\|_{L^\infty} \leq C\|f^\vartheta(n)\|_\infty, \text{ for all } t \in [0, 1].$$

Therefore,

$$\sup_{s \in [0, n+1]} \|f^\vartheta(s)\|_{L^\infty} \leq C\|f_0^\vartheta\|_{L^\infty} + C_{\vartheta, l}\|f_0\|_{2, \vartheta+l} + C\varepsilon \sup_{s \in [0, n+1]} \|f^\vartheta(s)\|_\infty.$$

Now, we choose small ε satisfying $C\varepsilon < 1/2$, and then absorb the last term on the RHS to the LHS. Thus, we obtain the first inequality of (5.17). The second inequality of (5.17) is a consequence of Proposition 2.1. \square

6. L^∞ TO HÖLDER ESTIMATE

6.1. Local Hölder estimate. In this subsection, we will derive a local Hölder estimate for (4.3). We redefine $Q_R(z_0) := (t_0 - R^2, t_0] \times B(x_0; R^3) \times B(v_0; R)$, $z_0 = (t_0, x_0, v_0)$, and $Q_R := Q_R((0, 0, 0))$.

Since we consider the local properties of the solution on the interior part, we can use the technique in [10] for our modified operator \tilde{A}_g . In this subsection, we assume that g satisfies the conditions in Lemma 2.4.

First, we introduce a De Giorgi-type lemma.

Lemma 6.1 (Lemma 13 in [10]). *Assume (2.4). Let $\hat{Q} := Q_{1/4}(0, 0, -1)$. For any (universal) constants $\delta_1 \in (0, 1)$ and $\delta_2 \in (0, 1)$ there exist $\nu > 0$ and $\vartheta \in (0, 1)$ (both universal) such that for any solution f of (4.3) in Q_2 with $|f| \leq 1$ and*

$$\begin{aligned} |\{f \geq 1 - \vartheta\} \cap Q_{1/4}| &\geq \delta_1 |Q_{1/4}|, \\ |\{f \leq 0\} \cap \hat{Q}| &\geq \delta_2 |\hat{Q}|, \end{aligned}$$

we have

$$|\{0 < f < 1 - \vartheta\} \cap B_1 \times B_1 \times (-2, 0]| \geq \nu.$$

Proof. The proof is exactly the same as [10]. We omit the proof. \square

Lemma 6.2 (Lemma 17 in [10]). *Assume (2.4). Let $\hat{Q} := Q_{1/4}(0, 0, -1)$ and f be a weak solution of (4.3) in Q_2 in the sense of Definition 3.1 with $|f| \leq 1$. If*

$$|\{f \leq 0\} \cap \hat{Q}| \geq \delta_2 |\hat{Q}|,$$

then

$$\sup_{Q_{1/8}} f \leq 1 - \lambda$$

for some $\lambda \in (0, 1)$, depending only on dimension and the eigenvalue of σ .

Proof. The proof is exactly the same as [10]. We omit the proof. \square

The following lemma can be derived by the previous lemma.

Lemma 6.3. *Assume (2.4). Let f be a weak solution of (4.3) in Q_2 in the sense of Definition 3.1 with $|f| \leq 1$. Then*

$$\sup_{Q_{1/8}} f - \inf_{Q_{1/8}} f \leq 2 - \lambda$$

for some $\lambda \in (0, 2)$, depending only on dimension and the eigenvalue of σ .

By the scaling argument, Q_2 and $Q_{1/8}$ can be replaced by Q_{2r} and $Q_{r/8}$.

Lemma 6.4. *Assume (2.4). Let f be a weak solution of (4.3) in Q_{2r} in the sense of Definition 3.1 with $|f| \leq 1$. For any subset $Q \subset \mathbb{R}^7$, define*

$$O_{\text{Scf}}^Q := \sup_{(t', x', v') \in Q} f(t', x', v') - \inf_{(t', x', v') \in Q} f(t', x', v'),$$

Then for every $r \leq 1$

$$O_{\text{Scf}}^{Q_{r/8}} \leq \left(1 - \frac{\lambda}{2}\right) O_{\text{Scf}}^{Q_{2r}}$$

for some $\lambda \in (0, 2)$, depending only on dimension and the eigenvalue of σ .

Proof. Define

$$\bar{F}(t, x, v) := \frac{2}{O_{\text{Scf}}^{Q_{2r}}} \left(f(r^2 t, r^3 x, r v) - \frac{\sup_{Q_{2r}} f + \inf_{Q_{2r}} f}{2} \right).$$

Then \bar{F} satisfies

$$\bar{F}_t + v \cdot \partial_x \bar{F} = \tilde{A}_g^r \bar{F}$$

$$\tilde{A}_g^r \bar{F}(t, x, v) := \nabla_v (\sigma_G(r^2 t, r^3 x, r v) \nabla_v \bar{F}(t, x, v)) + r a_g(r^2 t, r^3 x, r v) \cdot \nabla_v \bar{F}(t, x, v).$$

and then apply Lemma 6.3. \square

Now we establish the Hölder continuity at $v = 0$.

Lemma 6.5 (Hölder continuity near $v = 0$). *Assume (2.4). Let f be a weak solution of (4.3) in $\Omega_R(t_0, x_0, 0)$ in the sense of Definition 3.1. Then there exist a uniform constant $C > 0$ and a constant $\alpha \in (0, 1)$ depending only on dimension and the eigenvalue of σ_G such that*

$$\|f\|_{C^\alpha(Q_{R/128}(t_0, x_0, 0))} \leq \frac{C}{R^{3\alpha}} \|f\|_{L^\infty(Q_R(t_0, x_0, 0))},$$

for every $R < 1$.

Proof. We first prove

$$\sup_{(s, y, w) \in Q_{R/16}} \frac{|f(s, y, w) - f(0, 0, 0)|}{(|s| + |y| + |w|)^\alpha} \leq \frac{C}{R^{3\alpha}} \|f\|_{L^\infty(Q_R)}.$$

Define O_{Scf}^Q as in Lemma 6.4 and

$$\varphi(r) := r^{-\alpha_0} O_{\text{Scf}}^{Q_r},$$

where $\alpha_0 > 0$ can be chosen later. By Lemma 6.4,

$$\mathcal{O}_{Q_{r/16}}^{\text{cs}} f \leq \left(1 - \frac{\lambda}{2}\right) \mathcal{O}_{Q_r}^{\text{cs}} f. \quad (6.1)$$

Choose α_0 such that $16^{\alpha_0} \left(1 - \frac{\lambda}{2}\right) < 1$. Then by (6.1),

$$\begin{aligned} \varphi\left(\frac{r}{16}\right) &= r^{-\alpha_0} 16^{\alpha_0} \mathcal{O}_{Q_{r/4}(t,x,v)}^{\text{cs}} f \\ &\leq 16^{\alpha_0} \left(1 - \frac{\lambda}{2}\right) r^{-\alpha_0} \mathcal{O}_{Q_r}^{\text{cs}} f \\ &< \varphi(r). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \sup_{0 < r \leq R/16} \varphi(r) &\leq \sup_{\frac{R}{16} < r \leq R} \varphi(r) \\ &\leq 2 \frac{16^{\alpha_0}}{R^{\alpha_0}} \sup_{(t,x,v) \in Q_R} |f(t,x,v)|. \end{aligned} \quad (6.2)$$

If $(t,x,v) \in \partial Q_r$ then $|t| + |x| + |v| \geq r^3$. Therefore, for $3\alpha = \alpha_0$ and $r \leq R/16$, by (6.2)

$$\begin{aligned} \sup_{(s,y,w) \in Q_{R/16}} \frac{|f(s,y,w) - f(0,0,0)|}{(|s| + |y| + |w|)^\alpha} &= \sup_{(s,y,w) \in \partial Q_r, r \in (0, R/16)} \frac{|f(s,y,w) - f(0,0,0)|}{(|s| + |y| + |w|)^\alpha} \\ &\leq \sup_{(s,y,w) \in Q_r, r \in (0, R/16)} \frac{|f(s,y,w) - f(0,0,0)|}{r^{\alpha_0}} \\ &\leq \sup_{r \in (0, R/16)} \varphi(r) \\ &\leq \frac{C}{R^{\alpha_0}} \sup_{(t,x,v) \in Q_R} |f(t,x,v)|. \end{aligned} \quad (6.3)$$

Now we consider the general case. For any $(t_*, x_*, v_*) \in Q_{R/32}(t_0, x_0, 0)$, define the translated function

$$\begin{aligned} F(T, X, V) &= f(t, x, v), \\ T &= t - t_*, \quad X = x - x_* - T v_*, \quad V = v - v_*. \end{aligned}$$

Then F satisfies,

$$\partial_T F + V \cdot \nabla_X F = \nabla_V \cdot (\Sigma_G(t, x, v) \nabla_V F) + a_g(t, x, v) \cdot \nabla_V F.$$

Therefore, by (6.3),

$$\sup_{(s,y,w) \in Q_{R_1/16}} \frac{|F(s,y,w) - F(0,0,0)|}{(|s| + |y| + |w|)^\alpha} \leq \frac{C}{R_1^{\alpha_0}} \sup_{(t,x,v) \in Q_{R_1}} |F(t,x,v)|$$

for every $R_1 < 1$. Since $|v_*| \leq R/128$,

$$(t, x, v) \in Q_{R/64}(t_*, x_*, v_*) \text{ implies } (T, X, V) \in Q_{R/32}$$

and

$$(T, X, V) \in Q_{R/2} \text{ implies } (t, x, v) \in Q_R(t_*, x_*, v_*).$$

Therefore, by (6.3)

$$\begin{aligned}
& \sup_{(t,x,v) \in Q_{R/64}(t_*, x_*, v_*)} \frac{|f(t, x, v) - f(t_*, x_*, v_*)|}{(|t - t_*| + |x - x_*| + |v - v_*|)^\alpha} \\
& \leq (1 + |v_*|)^\alpha \sup_{(t,x,v) \in Q_{R/64}(t_*, x_*, v_*)} \frac{|f(t, x, v) - f(t_*, x_*, v_*)|}{((1 + |v_*|)|t - t_*| + |x - x_*| + |v - v_*|)^\alpha} \\
& \leq C \sup_{(T,X,V) \in Q_{R/32}} \frac{|F(T, X, V) - F(0, 0, 0)|}{((1 + |v_*|)|T| + |X + v_*T| + |V|)^\alpha} \\
& \leq C \sup_{(T,X,V) \in Q_{R/32}} \frac{|F(T, X, V) - F(0, 0, 0)|}{(|T| + |X| + |V|)^\alpha} \\
& \leq \frac{C}{R^{\alpha_0}} \sup_{(T,X,V) \in Q_{R/2}} |F(T, X, V)| \\
& \leq \frac{C}{R^{\alpha_0}} \sup_{(t,x,v) \in Q_R} |f(t, x, v)|.
\end{aligned}$$

So the proof is complete. \square

6.2. Global Hölder estimate. In this subsection, we will derive a Hölder continuity for the solution of (1.25). Let $f(t, x, v)$ be a weak solution of (1.25) in the sense of Definition 4.1. Then

$$\tilde{f}(t, x, v) := \begin{cases} f(t, x, v), & \text{if } t \geq 0, \\ f_0(x, v), & \text{if } -1 \leq t < 0, \end{cases}$$

satisfies

$$\tilde{f}_t + v \cdot \nabla_x \tilde{f} - \bar{A}_g \tilde{f} = \tilde{S}(t, x, v),$$

where \bar{A}_g and \bar{K}_g is defined as in (1.15), and (1.16),

$$\tilde{S}(t, x, v) = \begin{cases} (v \cdot \nabla_x - \bar{A}_{f_0})f_0(x, v), & \text{if } t \leq 0, \\ \bar{K}_g f(t, x, v), & \text{if } t > 0. \end{cases}$$

Since $U(t, s)$ is the solution operator of (4.3). Then f satisfies

$$f(t) = U(t, -1)f_0 + \int_{-1}^t U(t, s)\tilde{S}(s)ds.$$

First, we will obtain a uniform Hölder continuity of $U(t, s)f$. Finally, we will derive a uniform Hölder continuity of $f(t)$.

As a starting point, we introduce a technical lemma to obtain a uniform Hölder continuity of $U(t, s)f$.

Lemma 6.6. *Let $(t_*, x_*, v_*) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$, $N - 1/2 \leq |v_*| \leq N + 1/2$, $m > 9$ and O be an orthonormal matrix. Define*

$$D := \begin{bmatrix} (1 + |v_*|)^{-3/2} & 0 & 0 \\ 0 & (1 + |v_*|)^{-1/2} & 0 \\ 0 & 0 & (1 + |v_*|)^{-1/2} \end{bmatrix}, \quad (6.4)$$

$$X := D^{-1}O^T(x - v_*(t - t_*)), \quad X_* := D^{-1}O^T x_*, \quad V := D^{-1}O^T(v - v_*),$$

$$r_0 := (2 + N)^{-m}, \quad r_1 := (2 + N)^{-\frac{2m}{3} + \frac{5}{6}}, \quad r_2 := (2 + N)^{-\frac{4m}{9} + \frac{13}{18}}.$$

Then if $(t, x, v) \in Q_{r_0}(t_*, x_*, v_*)$, then $(t, X, V) \in Q_{r_1}(t_*, X_*, 0)$. Moreover, if $(t, X, V) \in Q_{128r_1}(t, X_*, 0)$, then $(t, x, v) \in Q_{128r_2}(t_*, x_*, v_*)$.

Proof. If $(t, x, v) \in Q_{r_0}(t_*, x_*, v_*)$, then

$$|t - t_*| \leq r_0^2 \leq r_1^2,$$

$$|X - X_*| = |D^{-1}O^T(x - x_* - v_*(t - t_*))|$$

$$\leq (2 + N)^{3/2}(r_0^3 + Nr_0^2)$$

$$\leq (2 + N)^{3/2}((2 + N)^{-3m} + N(2 + N)^{-2m})$$

$$\leq (2 + N)^{3/2}(2 + N)^{1-2m}$$

$$\leq r_1^3,$$

and

$$|V| = |D^{-1}O^T(v - v_*)|$$

$$\leq (2 + N)^{3/2}r_0$$

$$\leq r_1.$$

Conversely, if $(t, X, V) \in Q_{128r_1}(t, X_*, 0)$, then

$$|t - t_*| \leq (128r_1)^2 \leq (128r_2)^2$$

and

$$|v - v_*| = |ODV| \leq (1/2 + N)^{-1/2}r_1 \leq 128r_2.$$

Since $128r_1 \leq 1$ and $(1/2 + N)^{-1/2}(1 + N) \leq 128(2 + N)^{1/2}$, we have

$$|x - x_*| = |OD(X - X_*) + v_*(t - t_*)|$$

$$\leq (1/2 + N)^{-1/2}(r_1^3 + Nr_1^2)$$

$$\leq (1/2 + N)^{-1/2}(1 + N)r_1^2$$

$$\leq (128)^3(2 + N)^{\frac{1}{2} - \frac{4m}{3} + \frac{5}{3}}$$

$$\leq (128r_2)^3.$$

So the proof is complete. \square

Lemma 6.7 (Uniform Hölder for (4.3)). *Assume (2.4). Let f be a solution of (4.3) in $Q_1(t_0, x_0, v_0)$. Then there exist $\vartheta > 0$, $\vartheta_0 > 0$, C_ϑ , and $\alpha \in (0, 1)$ depending only on dimension such that*

$$\sup_{(t,x,v),(t',x',v') \in Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v) - f(t',x',v')|}{(|t - t'| + |x - x'| + |v - v'|)^\alpha} \leq C\|f\|_{\infty,\vartheta} \leq C_\vartheta\|f_0\|_{\infty,\vartheta+\vartheta_0}. \quad (6.5)$$

Proof. By the integration by parts,

$$\begin{aligned}
a_g \cdot \nabla_v f &= - \left\{ \phi^{ij} * [v_i \mu^{1/2} g] \right\} \partial_j f - \left\{ \phi^{ij} * [\mu^{1/2} \partial_j g] \right\} \partial_i f \\
&= - \left(\left\{ \phi^{ij} * [v_i \mu^{1/2} g] \right\} + \left\{ \phi^{ij} * [\mu^{1/2} \partial_i g] \right\} \right) \partial_j f \\
&= - \left(2 \left\{ \phi^{ij} * [v_i \mu^{1/2} g] \right\} + \left\{ \phi^{ij} * [\partial_i (\mu^{1/2} g)] \right\} \right) \partial_j f \\
&= - \left(2 \left\{ \phi^{ij} * [v_i \mu^{1/2} g] \right\} + \left\{ \partial_i \phi^{ij} * [\mu^{1/2} g] \right\} \right) \partial_j f \\
&= -2v \cdot (\sigma_{\sqrt{\mu}g} \nabla_v f) - \partial_i \sigma_{\sqrt{\mu}g}^{ij} \partial_j f.
\end{aligned}$$

Let $N := |v_0|$.

To obtain (6.5), we split the proof in two cases; $|(t, x, v) - (t', x', v')| \leq (2 + N)^{-3m}$ or $|(t, x, v) - (t', x', v')| > (2 + N)^{-3m}$ for some $m > 0$ to be determined later. For the first case, we will consider a new center $(t_*, x_*, v_*) \in Q_1$, such that $(t, x, v), (t', x', v') \in Q_{(2+N)^{-m}}(t_*, x_*, v_*)$. Note that $N - 1/2 \leq |v_*| \leq N + 1/2$.

Therefore, it is enough to prove that for every $(t_*, x_*, v_*) \in Q_1(t_0, x_0, v_0)$,

$$\sup_{(t,x,v),(t',x',v') \in Q_{(2+N)^{-m}}(t_*,x_*,v_*) \cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v) - f(t',x',v')|}{(|t-t'| + |x-x'| + |v-v'|)^\alpha} \leq C \|f\|_{\infty, \vartheta} \quad (6.6)$$

and

$$\sup_{\substack{(t,x,v),(t',x',v') \in Q_1(t_0,x_0,v_0) \\ |t-t'| + |x-x'| + |v-v'| > (2+N)^{-3m}}} \frac{|f(t,x,v) - f(t',x',v')|}{(|t-t'| + |x-x'| + |v-v'|)^\alpha} \leq C \|f\|_{\infty, \vartheta}. \quad (6.7)$$

We first focus on (6.6). Consider the the following translation

$$\bar{f}(t, y, w) := f(t, x, v),$$

where $x = y + v_*(t - t_*)$, $v = v_* + w$. Then it is easy to check that \bar{f} satisfies

$$\partial_t \bar{f} + w \cdot \nabla_y \bar{f} = \nabla_w \cdot (\bar{\sigma}_G \nabla_w \bar{f}) + (v_* + w) \cdot (\bar{\sigma}_{\sqrt{\mu}g} \nabla_w \bar{f}) - \sum_{ij} \partial_i \bar{\sigma}_{\sqrt{\mu}g}^{ij} \partial_j \bar{f},$$

where $\bar{\sigma}_G(t, y, w) := \sigma_G(t, x, v)$, $\bar{\sigma}_\mu(t, y, w) := \sigma_\mu(t, x, v)$, and $\bar{\sigma}_{\sqrt{\mu}g}(t, y, w) := \sigma_{\sqrt{\mu}g}(t, x, v)$. Let O be an orthonormal constant matrix which will be determined later. Next consider

$$\tilde{f}(t, \xi, \nu) := \bar{f}(t, y, w),$$

where $y = O\xi$, $w = O\nu$. Then we have

$$\partial_t \tilde{f}(t, y, w) = \partial_t \tilde{f}(t, \xi, \nu).$$

$$\begin{aligned}
w \cdot \nabla_y \tilde{f}(t, y, w) &= \sum_i w_i \partial_{y_i} (\tilde{f}(t, \xi, \nu)) \\
&= \sum_{i,k} w_i \partial_{\xi_k} \tilde{f}(t, \xi, \nu) \frac{\partial \xi_k}{\partial y_i} \\
&= \sum_{i,k} O_{ik} w_i \partial_{\xi_k} \tilde{f}(t, \xi, \nu) \\
&= \sum_k (O^T w)_k \cdot \partial_{\xi_k} \tilde{f}(t, \xi, \nu) \\
&= \nu \cdot \nabla_\xi \tilde{f}(t, \xi, \nu).
\end{aligned} \tag{6.8}$$

where O_{ik} is the i, k component of O . We use the following formula to derive the third equality in (6.8)

$$\frac{\partial \xi_k}{\partial y_i} = \frac{\partial \sum_l O_{lk} y_l}{\partial y_i} = O_{ik}.$$

Similarly,

$$\begin{aligned}
\bar{\sigma}_G(t, y, w) \nabla_w \tilde{f}(t, y, w) &= \sum_j \bar{\sigma}_G^{ij}(t, y, w) \partial_{w_j} \tilde{f}(t, \xi, \nu) \\
&= \sum_{j,k} \bar{\sigma}_G^{ij}(t, y, w) O_{jk} \partial_{\nu_k} \tilde{f}(t, \xi, \nu).
\end{aligned} \tag{6.9}$$

Define $\tilde{\sigma}_G(t, \xi, \nu) := O^T \bar{\sigma}_G(t, y, w) O$. Note that

$$O \tilde{\sigma}_G(t, \xi, \nu) = \bar{\sigma}_G(t, y, w) O. \tag{6.10}$$

Then by (6.9) and (6.10), we have

$$\begin{aligned}
\nabla_w \cdot (\bar{\sigma}_G(t, y, w) \nabla_w \tilde{f}(t, y, w)) &= \sum_{i,j} \partial_{w_i} (\bar{\sigma}_G^{ij}(t, y, w) \partial_{w_j} \tilde{f}(t, \xi, \nu)) \\
&= \sum_{i,j,k} \partial_{w_i} (O_{jk} \bar{\sigma}_G^{ij}(t, y, w) \partial_{\nu_k} \tilde{f}(t, \xi, \nu)) \\
&= \sum_{i,j,k} \partial_{w_i} (O_{ij} \tilde{\sigma}_G^{jk}(t, \xi, \nu) \partial_{\nu_k} \tilde{f}(t, \xi, \nu)) \\
&= \sum_{i,j,k,l} O_{il} O_{ij} \partial_{\nu_*} (\tilde{\sigma}_G^{jk}(t, \xi, \nu) \partial_{\nu_k} \tilde{f}(t, \xi, \nu)) \\
&= \sum_{k,l} \partial_{\nu_*} (\tilde{\sigma}_G^{lk}(t, \xi, \nu) \partial_{\nu_k} \tilde{f}(t, \xi, \nu)) \\
&= \nabla_\nu \cdot (\tilde{\sigma}_G(t, \xi, \nu) \nabla_\nu \tilde{f}(t, \xi, \nu)).
\end{aligned}$$

In the next to the last equality, we use $O^T O = I$. Similarly, define $\tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, \nu) := O^T \tilde{\sigma}_{\sqrt{\mu}g}(t, y, w) O$, then

$$\begin{aligned} v_* \cdot (\tilde{\sigma}_{\sqrt{\mu}g}(t, y, w) \nabla_w \tilde{f}(t, y, w)) &= v_* \cdot (O \tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, \nu) \nabla_\nu \tilde{f}(t, \xi, \nu)) \\ &= (O^T v_*) \cdot (\tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, \nu) \nabla_\nu \tilde{f}(t, \xi, \nu)) \\ &= \nu_* \cdot (\tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, \nu) \nabla_\nu \tilde{f}(t, \xi, \nu)), \end{aligned}$$

where $\nu_* = O^T v_*$,

$$\begin{aligned} w \cdot (\tilde{\sigma}_{\sqrt{\mu}g}(t, y, w) \nabla_w \tilde{f}(t, y, w)) &= w \cdot (O \tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, \nu) \nabla_\nu \tilde{f}(t, \xi, \nu)) \\ &= (O^T w) \cdot (\tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, \nu) \nabla_\nu \tilde{f}(t, \xi, \nu)) \\ &= \nu \cdot (\tilde{\sigma}_{\sqrt{\mu}g}(t, \xi, \nu) \nabla_\nu \tilde{f}(t, \xi, \nu)), \end{aligned}$$

and

$$\begin{aligned} \sum_{ij} \partial_{w_i} \tilde{\sigma}_{\sqrt{\mu}g}^{ij}(t, y, w) \partial_{w_j} \tilde{f}(t, y, w) &= \sum_{ijk} \partial_{w_i} \tilde{\sigma}_{\sqrt{\mu}g}^{ij}(t, y, w) \partial_{w_j} \tilde{f}(t, \xi, \nu) \\ &= \sum_{ijk} \partial_{w_i} \tilde{\sigma}_{\sqrt{\mu}g}^{ij}(t, y, w) O_{jk} \partial_{\nu_k} \tilde{f}(t, \xi, \nu) \\ &= \sum_{ijkl} O_{il} \partial_{\nu_*} \tilde{\sigma}_{\sqrt{\mu}g}^{ij}(t, y, w) O_{jk} \partial_{\nu_k} \tilde{f}(t, \xi, \nu) \\ &= \sum_{lk} \partial_{\nu_*} \tilde{\sigma}_{\sqrt{\mu}g}^{lk}(t, \xi, \nu) \partial_{\nu_k} \tilde{f}(t, \xi, \nu). \end{aligned}$$

Therefore \tilde{f} satisfies

$$\partial_t \tilde{f} + \nu \cdot \nabla_\xi \tilde{f} = \nabla_\nu \cdot (\tilde{\sigma}_G \nabla_\nu \tilde{f}) + (\nu_* + \nu) \cdot (\tilde{\sigma}_{\sqrt{\mu}g} \nabla_\nu \tilde{f}) - \sum_{lk} \partial_l \tilde{\sigma}_{\sqrt{\mu}g}^{lk} \partial_k \tilde{f}.$$

We split $\tilde{\sigma}_G(t, \xi, \nu)$ in three parts.

$$\begin{aligned} \tilde{\sigma}_G(t, \xi, \nu) &= O^T \tilde{\sigma}_\mu(0) O \\ &\quad + O^T (\tilde{\sigma}_\mu(w) - \tilde{\sigma}_\mu(0)) O \\ &\quad + O^T \tilde{\sigma}_{\sqrt{\mu}g}(t, y, w) O \\ &= \tilde{\sigma}_1 + \tilde{\sigma}_2 + \tilde{\sigma}_3. \end{aligned}$$

Choose orthonormal vectors $o_1 = v_*/|v_*|$, o_2 , o_3 and

$$O := \begin{bmatrix} o_1 & o_2 & o_3 \end{bmatrix}.$$

Note that

$$\nu_* = O^T v_* = \begin{bmatrix} |v_*| \\ 0 \\ 0 \end{bmatrix}.$$

Moreover $\bar{\sigma}_\mu(0)$ has a simple eigenvalue $\lambda_1(v_*)$ associated with the vector v_* , and a double eigenvalue $\lambda_2(v_*)$ associated with v^\perp . Therefore,

$$\tilde{\sigma}_1 = \begin{bmatrix} \lambda_1(v_*) & 0 & 0 \\ 0 & \lambda_2(v_*) & 0 \\ 0 & 0 & \lambda_2(v_*) \end{bmatrix}.$$

Note that $\lambda_1(v_*)$ and $\lambda_2(v_*)$ satisfy

$$\begin{aligned} \frac{1}{C}(1+N)^{-3} &\leq \lambda_1(v_*) \leq C(1+N)^{-3}, \\ \frac{1}{C}(1+N)^{-1} &\leq \lambda_2(v_*) \leq C(1+N)^{-1}. \end{aligned} \tag{6.11}$$

Since $\partial_{v_k}(\sigma_\mu)^{ij}(v) \leq C(1+|v|)^{-2}$, by the mean value theorem,

$$|(\bar{\sigma}_\mu)^{ij}(w) - (\bar{\sigma}_\mu(0))| \leq C(1+N)^{-3+1}(2+N)^{-m}.$$

Therefore,

$$|(\tilde{\sigma}_2)^{ij}| \leq C(1+N)^{-2}(2+N)^{-m}.$$

Define

$$D_u(\nu, \nu'; v) := \nu^T \sigma_u(v) \nu'.$$

Then we can easily check that

$$|D_u(\nu, \nu'; v)| \leq |D_u(\nu, \nu; v)|^{1/2} |D_u(\nu', \nu'; v)|^{1/2}.$$

Since $|v - v_*| < (2+N)^{-m}$ and $o_1 = v_*/|v_*|$, we have

$$\begin{aligned} |(I - P_v)o_1| &= \frac{|(I - P_v)v_*|}{|v_*|} \\ &= \frac{|-v + v_* + v - P_v v_*|}{|v_*|} \\ &\leq \frac{|v - v_*| - |v - P_v v_*|}{|v_*|} \\ &= \frac{|v - v_*| - |P_v(v - v_*)|}{|v_*|} \\ &= 2 \frac{|v - v_*|}{|v_*|} \leq C(2+N)^{-m}. \end{aligned} \tag{6.12}$$

Note that

$$(\tilde{\sigma}_3)^{ij} = o_i^T \sigma_{\sqrt{\mu}g}(v) o_j. \tag{6.13}$$

Therefore, by (2.5),

$$\begin{aligned} |(\tilde{\sigma}_3)^{11}| &= |D_{\sqrt{\mu}g}(o_1; v)| \\ &\leq C\|g\|_\infty ((1+|v|)^{-3}|P_v o_1|^2 + (1+|v|)^{-1}|(I - P_v)o_1|^2) \\ &\leq C\|g\|_\infty ((1+N)^{-3} + (1+N)^{-1}(2+N)^{-2m}) \\ &\leq C\|g\|_\infty (1+N)^{-3}, \end{aligned} \tag{6.14}$$

and for $(i, j) \in \{(1, 2), (1, 3), (2, 1), (3, 1)\}$,

$$\begin{aligned} |(\tilde{\sigma}_3)^{ij}| &\leq |D_{\sqrt{\mu}g}(o_1; v)|^{1/2} |D_{\sqrt{\mu}g}(o_k; v)|^{1/2} \\ &\leq C\|g\|_\infty (1+N)^{-3/2} \left((1+|v|)^{-3} |P_v o_k|^2 + (1+|v|)^{-1} |(I - P_v) o_k|^2 \right)^{1/2} \\ &\leq C\|g\|_\infty (1+N)^{-3/2} (1+N)^{-1/2} \\ &= C\|g\|_\infty (1+N)^{-2}, \end{aligned}$$

where $k = 2$ or 3 . Finally, for $i, j = 2$ or 3 ,

$$\begin{aligned} |(\tilde{\sigma}_3)^{ij}| &\leq |D_{\sqrt{\mu}g}(o_i; v)|^{1/2} |D_{\sqrt{\mu}g}(o_j; v)|^{1/2} \\ &= C\|g\|_\infty \left((1+|v|)^{-3} |P_v o_i|^2 + (1+|v|)^{-1} |(I - P_v) o_i|^2 \right)^{1/2} \\ &\quad \times \left((1+|v|)^{-3} |P_v o_j|^2 + (1+|v|)^{-1} |(I - P_v) o_j|^2 \right)^{1/2} \\ &\leq C\|g\|_\infty (1+N)^{-1}. \end{aligned}$$

Finally, consider the dilation matrix D as in (6.4) and the dilated function

$$F(t, X, V) := \tilde{f}(t, \xi, \nu),$$

where $\xi = DX$, $\nu = DV$. Then we can easily check that F satisfies

$$\partial_t F + V \cdot \nabla_X F = \nabla_V \cdot (\Sigma \nabla_V F) + (\nu_l + \nu)^T D \Sigma_3 \nabla_V F + \sum_{lk} \partial_{V_l} \Sigma_3^{lk} \partial_{V_k} F,$$

where $\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3$, $\Sigma_i(t, X, V) = D^{-1} \tilde{\sigma}_i(t, \xi, \nu) D^{-1}$ for $i = 1, 2, 3$. Then by (6.11), we have

$$\frac{1}{C} \leq (\Sigma_1)^{ii} \leq C, \quad (\Sigma_1)^{ij} = 0 \text{ for } i \neq j$$

and

$$\begin{aligned} |(\Sigma_2)^{ij}| &\leq C(1+N)(2+N)^{-m}, \\ |(\Sigma_3)^{ij}| &\leq C\|g\|_\infty. \end{aligned}$$

Moreover, since $|\nu| \leq (2+N)^{-m}$,

$$\begin{aligned} |D(\nu_* + \nu)| &\leq |D\nu_*| + |D\nu| \\ &\leq (1+N)^{-1/2} + C(1+N)^{-1/2}(2+N)^{-m}, \end{aligned}$$

and since $\partial_{\nu_k} \tilde{\sigma}_3(\nu) \leq C\|g\|_\infty (1+N)^{-2}$, we have

$$\begin{aligned} |\partial_{V_l} \Sigma_3^{lk}| &= |d_l^{-1} d_k^{-1} \partial_{V_l} \tilde{\sigma}_3(\nu)| \\ &\leq d_k^{-1} |\partial_{\nu_l} \tilde{\sigma}_3(\nu)| \\ &\leq \|g\|_\infty (1+N)^{-1/2}, \end{aligned}$$

where d_k 's are the k th diagonal element of D . Choose $m > 4$ such that $|(\Sigma_2)^{ij}|, |(\nu_* + \nu)^T D \Sigma_3| \leq \varepsilon \ll 1$. If $\|g\|_\infty \leq \varepsilon$, then any eigenvalue of Σ is bounded above and below uniformly in N . Therefore,

by Lemma 6.5, there exist a constant $C > 0$ uniformly in N and a constant $\alpha \in (0, 1)$, depending only on dimension such that

$$\begin{aligned} & \sup_{(t,X,V),(t',X',V') \in Q_{r_1}(t_*,X_*,0)} \frac{|F(t,X,V) - F(t',X',V')|}{(|t-t'| + |X-X'| + |V-V'|)^\alpha} \\ & \leq \frac{C}{(r_1)^{3\alpha}} \|F\|_{L^\infty(Q_{128r_1}(t_*,X_*,0))}, \end{aligned} \quad (6.15)$$

where $X_* = D^{-1}\xi_*$, $\xi_* = O^T y_l$, $y_l = x_* - v_* t_*$ and r_1 is defined as in Lemma 6.5. Note that

$$\begin{aligned} & \frac{1}{(1+N)^\alpha} \sup_{(t,x,v),(t',x',v') \in Q_{r_0}(t_*,x_*,v_*) \cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v) - f(t',x',v')|}{(|t-t'| + |x-x'| + |v-v'|)^\alpha} \\ & \leq \sup_{(t,x,v),(t',x',v') \in Q_{r_0}(t_*,x_*,v_*) \cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v) - f(t',x',v')|}{((2+N)|t-t'| + |x-x'| + |v-v'|)^\alpha}, \end{aligned} \quad (6.16)$$

where r_0 and r_1 are defined as in Lemma 6.6. Moreover, we have

$$\begin{aligned} & (2+N)|t-t'| + |x-x'| + |v-v'| \\ & = (2+N)|t-t'| + |ODX + v_*(t-t_*) - (ODX' + v_*(t'-t_*))| + |ODV + v_* - (ODV' + v_*)| \\ & \geq (2+N)|t-t'| + |OD(X-X')| - |v_*||t-t'| + |OD(V-V')| \\ & \geq |t-t'| + |OD(X-X')| + |OD(V-V')| \\ & \geq (1+N)^{-3/2}(|t-t'| + |(X-X')| + |(V-V')|). \end{aligned} \quad (6.17)$$

By (6.15), (6.16), (6.17), and Lemma 6.6, we have

$$\begin{aligned} & \sup_{(t,x,v),(t',x',v') \in Q_{r_0}(t_*,x_*,v_*) \cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v) - f(t',x',v')|}{((2+N)|t-t'| + |x-x'| + |v-v'|)^\alpha} \\ & \leq \sup_{(t,X,V),(t',X',V') \in Q_{r_1}(t_*,X_*,0)} \frac{|F(t,X,V) - F(t',X',V')|}{(1+N)^{-3\alpha/2}(|t-t'| + |(X-X')| + |(V-V')|)^\alpha}. \end{aligned} \quad (6.18)$$

Combine (6.15), (6.16), and (6.18). Then we have

$$\begin{aligned} & \sup_{(t,x,v),(t',x',v') \in Q_{r_0}(t_*,x_*,v_*) \cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v) - f(t',x',v')|}{(|t-t'| + |x-x'| + |v-v'|)^\alpha} \\ & \leq (2+N)^\alpha \sup_{(t,x,v),(t',x',v') \in Q_{r_0}(t_*,x_*,v_*) \cap Q_1(t_0,x_0,v_0)} \frac{|f(t,x,v) - f(t',x',v')|}{((2+N)|t-t'| + |x-x'| + |v-v'|)^\alpha} \\ & \leq \frac{(2+N)^\alpha}{(1+N)^{-3\alpha/2}} \sup_{(t,X,V),(t',X',V') \in Q_{r_1}(t_*,X_*,0)} \frac{|F(t,X,V) - F(t',X',V')|}{(|t-t'| + |X-X'| + |(V-V')|)^\alpha} \\ & \leq \frac{C(2+N)^\alpha}{(1+N)^{-3\alpha/2}(r_1)^{3\alpha}} \|F\|_{L^\infty(Q_{128r_1}(t_*,X_*,0))}. \end{aligned} \quad (6.19)$$

By the Lemma 6.6,

$$\|F\|_{L^\infty(Q_{128r_1}(t_*,X_*,0))} \leq \|f\|_{L^\infty(Q_{128r_2}(t_*,x_*,v_*))}. \quad (6.20)$$

Choose $m > 9$ such that $128r_2 < 1$. Then we have

$$\begin{aligned} \|f\|_{L^\infty(Q_{128r_2}(t_*, x_*, v_*))} &\leq C(1+N)^{-\vartheta} \|(1+|v|)^\vartheta f\|_{L^\infty(Q_{128r_2}(t_*, x_*, v_*))} \\ &\leq C(1+N)^{-\vartheta} \|(1+|v|)^\vartheta f\|_{L^\infty}, \end{aligned} \quad (6.21)$$

for every $\vartheta > 0$. Finally combining (6.19) - (6.21), we have

$$\begin{aligned} &\sup_{(t,x,v),(t',x',v') \in Q_{r_0}(t_*, x_*, v_*) \cap Q_1(t_0, x_0, v_0)} \frac{|f(t, x, v) - f(t', x', v')|}{(|t-t'| + |x-x'| + |v-v'|)^\alpha} \\ &\leq \frac{C(1+N)^{\alpha-\vartheta}}{(1+N)^{-3\alpha/2}(r_1)^{3\alpha}} \|(1+|v|)^\vartheta f\|_{L^\infty} \\ &\leq C(1+N)^{\alpha-\vartheta+\frac{3\alpha}{2}+2m\alpha-\frac{5\alpha}{2}} \|(1+|v|)^\vartheta f\|_{L^\infty} \\ &= C(1+N)^{-\vartheta+2m\alpha} \|(1+|v|)^\vartheta f\|_{L^\infty}, \end{aligned} \quad (6.22)$$

where $\vartheta > 0$ will be determined later.

To prove (6.7),

$$\begin{aligned} &\sup_{\substack{(t,x,v),(t',x',v') \in Q_1(t_0, x_0, v_0) \\ |t-t'|+|x-x'|+|v-v'| > (2+N)^{-3m}}} \frac{|f(t, x, v) - f(t', x', v')|}{(|t-t'| + |x-x'| + |v-v'|)^\alpha} \\ &\leq 2(2+N)^{3\alpha m} \|f\|_{L^\infty(Q_1(t_0, x_0, v_0))} \\ &\leq 2C(1+N)^{3\alpha m-\vartheta} \|(1+|v|)^\vartheta f\|_{L^\infty(Q(t_0, x_0, v_0; 1))} \\ &\leq 2C(1+N)^{3\alpha m-\vartheta} \|(1+|v|)^\vartheta f\|_{L^\infty}. \end{aligned} \quad (6.23)$$

Now choose

$$\vartheta > 3\alpha m. \quad (6.24)$$

Then from (6.22) - (6.24), we prove (6.6) and (6.7). Therefore, we have

$$\sup_{(t,x,v),(t',x',v') \in Q_1(t_0, x_0, v_1)} \frac{|f(t, x, v) - f(t', x', v')|}{(|t-t'| + |x-x'| + |v-v'|)^\alpha} \leq C\|f\|_{\infty, \vartheta}.$$

By Theorem 5.13, we have (6.5). □

Now we will prove Theorem 1.6.

Proof of Theorem 1.6. Since f satisfies (1.25), we have

$$f_t + v \cdot \nabla_x f - \bar{A}_g f = \bar{K}_g f. \quad (6.25)$$

Define

$$\tilde{f}(t, x, v) = \begin{cases} f(t, x, v), & \text{if } t \geq 0, \\ f_0(x, v), & \text{if } -1 \leq t < 0. \end{cases}$$

Consider $\tilde{S}(t, x, v) = (\partial_t + v \cdot \nabla_x - \bar{A}_g) \tilde{f}$. Then for $t \leq 0$,

$$\begin{aligned} \tilde{S}(t, x, v) &= (\partial_t + v \cdot \nabla_x - \bar{A}_g) \tilde{f} \\ &= (v \cdot \nabla_x - \bar{A}_{f_0}) f_0(x, v) \\ &= -f_{0t}, \end{aligned}$$

where f_{0t} was defined in Theorem 1.3. Since f is a weak solution of (1.25) in the sense of Definition 4.1, for $t > 0$, $\tilde{S}(t, x, v) = \bar{K}_g f(t, x, v)$. Thus, \tilde{f} satisfies,

$$\tilde{f}_t + v \cdot \nabla_x \tilde{f} - \bar{A}_g \tilde{f} = \tilde{S}(t, x, v).$$

Since $U(t, s)$ is the solution operator for $\partial_t + v \cdot \nabla_x - \tilde{A}_g = 0$. Then \tilde{f} satisfies

$$\tilde{f}(t) = U(t, -1)f_0 + \int_{-1}^t U(t, s)\tilde{S}(s)ds.$$

Let $0 < \bar{\varepsilon} \ll 1$ be given. Note that by Lemma 6.5, there exists $\alpha > 0$ such that $U(t, -1)\tilde{f}(-1)$ is uniformly Hölder continuous on $(0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3$.

For every $0 \leq t_2 \leq t_1$, $|t_1 - t_2| + |x_2 - x_1| + |v_2 - v_1| = \bar{\varepsilon}$,

$$\begin{aligned} & |f(t_1, x_1, v_1) - f(t_2, x_2, v_2)| \\ & \leq |(U(t_1, -1)f_0)(x_1, v_1) - (U(t_2, -1)f_0)(x_2, v_2)| \\ & \quad + \left| \int_{t_2}^{t_1} (U(t_1, s)\tilde{S}(s))(x_1, v_1)ds \right| \\ & \quad + \left| \int_{t_2 - \bar{\varepsilon}^\alpha}^{t_2} ((U(t_1, s)\tilde{S}(s))(x_1, v_1) - (U(t_2, s)\tilde{S}(s))(x_2, v_2))ds \right| \\ & \quad + \left| \int_{-1}^{t_2 - \bar{\varepsilon}^\alpha} ((U(t_1, s)\tilde{S}(s))(x_1, v_1) - (U(t_2, s)\tilde{S}(s))(x_2, v_2))ds \right| \\ & \leq (I) + (II) + (III) + (IV). \end{aligned}$$

By Lemma 6.7, there exist ϑ , l , C and $C_{\vartheta, l}$ such that

$$(I) \leq (C\|f_0\|_{\infty, \vartheta} + C_{\vartheta, l}\|f_0\|_{2, \vartheta+l})\bar{\varepsilon}^\alpha.$$

By Lemma 3.7, (1.39), and (5.13), there exists l_0 such that

$$(II) \leq \bar{\varepsilon} \sup_{s>0} \|\tilde{K}_g f(s)\|_{\infty} \leq C\bar{\varepsilon} \sup_{s>0} \|f(s)\| \leq C\bar{\varepsilon}(\|f_0\|_{\infty} + \|f_0\|_{2, l_0}).$$

Note that for $s \leq 0$,

$$\|\tilde{S}(s)\|_{\infty, \vartheta} \leq C\|f_{0t}\|_{\infty, \vartheta}, \quad (6.26)$$

$$\|\tilde{S}(s)\|_{2, \vartheta} \leq C\|f_{0t}\|_{2, \vartheta}, \quad (6.27)$$

and for $s \geq 0$, by Lemma 2.9 and Theorem 1.4, 1.5 there exists l_0 such that

$$\|\tilde{S}(s)\|_{\infty, \vartheta} \leq C(1+s)^{-2}(\|f_0\|_{2, \vartheta+l_0} + \|f_0\|_{\infty, \vartheta+l_0}) \quad (6.28)$$

and

$$\|\tilde{S}(s)\|_{2, \vartheta+l} \leq C(1+s)^{-2}(\|f_0\|_{2, \vartheta+l_0} + \|f_0\|_{\infty, \vartheta+l_0}). \quad (6.29)$$

Therefore, by (6.26) and (6.29), we have

$$\begin{aligned} (III) & \leq \bar{\varepsilon}^\alpha \sup_{s \in (t_2 - \bar{\varepsilon}^\alpha, t_2)} \|\tilde{S}(s)\|_{\infty} \\ & \leq C\bar{\varepsilon}^\alpha(\|f_{0t}\|_{\infty} + \|f_0\|_{2, l_0} + \|f_0\|_{\infty, l_0}). \end{aligned}$$

For $-1 \leq s \leq t_2 - \bar{\varepsilon}^\alpha$, we have

$$\begin{aligned} & |t_1 - (t_2 + 1 - \bar{\varepsilon}^\alpha)| + |x_1 - x_2| + |v_1 - v_2| \\ &= |t_1 - t_2 - 1 + \bar{\varepsilon}^\alpha| + |x_1 - x_2| + |v_1 - v_2| \\ &\leq 1 - \bar{\varepsilon}^\alpha + |t_1 - t_2| + |x_1 - x_2| + |v_1 - v_2| \\ &\leq 1, \end{aligned}$$

$$|t_2 - (t_2 + 1 - \bar{\varepsilon}^\alpha)| + |x_2 - x_2| + |v_2 - v_2| = 1 - \bar{\varepsilon}^\alpha \leq 1.$$

Therefore,

$$(t_i, x_i, v_i) \in Q_1(t_2 + 1 - \bar{\varepsilon}^\alpha, x_2, v_2) \subset (s, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3,$$

for each $i = 1, 2$. Therefore by Lemma 6.7, there exist ϑ, l, C and $C_{\vartheta, l}$ such that

$$\begin{aligned} & (U(t_1, s)\tilde{S}(s))(x_1, v_1) - (U(t_2, s)\tilde{S}(s))(x_2, v_2) \\ & \leq (C\|\tilde{S}(s)\|_{\infty, \vartheta} + C_{\vartheta, l}\|\tilde{S}(s)\|_{2, \vartheta+l})\bar{\varepsilon}^\alpha, \end{aligned}$$

for $-1 \leq s \leq t_2 - \bar{\varepsilon}^\alpha$. Therefore, by Lemma 6.7, (6.26), (6.28), and (6.27), (6.29), we have

$$\begin{aligned} (IV) & \leq \left| \int_{-1}^0 ((U(t_1, s)\tilde{S}(s))(x_1, v_1) - (U(t_2, s)\tilde{S}(s))(x_2, v_2)) ds \right| \\ & \quad + \left| \int_0^{t_2 - \bar{\varepsilon}^\alpha} ((U(t_1, s)\tilde{S}(s))(x_1, v_1) - (U(t_2, s)\tilde{S}(s))(x_2, v_2)) ds \right| \\ & \leq C\bar{\varepsilon}^\alpha \left(\sup_{s \in [-1, 0]} (\|\tilde{S}(s)\|_{\infty, \vartheta} + \|\tilde{S}(s)\|_{2, \vartheta+l}) + \int_0^\infty \|\tilde{S}(s)\|_{\infty, \vartheta} + \|\tilde{S}(s)\|_{2, \vartheta+l} ds \right) \\ & \leq C\bar{\varepsilon}^\alpha \left(\|f_{0t}\|_{\infty, \vartheta} + \|f_{0t}\|_{2, \vartheta+l} + (\|f_0\|_{2, \vartheta+l_0} + \|f_0\|_{\infty, \vartheta+l_0}) \int_0^\infty (1+s)^{-2} ds \right) \\ & \leq C\bar{\varepsilon}^\alpha (\|f_{0t}\|_{\infty, \vartheta} + \|f_{0t}\|_{2, \vartheta+l} + \|f_0\|_{2, \vartheta+l_0} + \|f_0\|_{\infty, \vartheta+l_0}). \end{aligned}$$

Now we update ϑ to $\vartheta + 2 + \max\{l, l_0\}$. Then, by Proposition 2.1, we have

$$\begin{aligned} |f(t_1, x_1, v_1) - f(t_2, x_2, v_2)| & \leq (I) + (II) + (III) + (IV) \\ & \leq C\bar{\varepsilon}^\alpha (\|f_{0t}\|_{\infty, \vartheta} + \|f_0\|_{\infty, \vartheta}). \end{aligned}$$

Thus we complete the proof. \square

7. HOLDER ESTIMATE AND S_p BOUND

Let f be a weak solution of (1.25) in the sense of Definition 4.1. Define

$$\bar{f}(t, x, v) = \begin{cases} f(t, x, v), & \text{if } t \geq 0 \\ f_0(x, v), & \text{if } -1 \leq t < 0. \end{cases}$$

Then \bar{f} satisfies

$$\begin{aligned} \partial_t \bar{f} + v \cdot \nabla_x \bar{f} - \sigma_G^{ij} \partial_{v_i, v_j} \bar{f} &= \begin{cases} -\partial_{v_i} \sigma_G^{ij} \partial_{v_j} f + a_g \cdot \nabla_v f + K_1 f + J_g f, & \text{if } t \geq 0, \\ (v \cdot \nabla_x - \sigma_{\mu+\mu^{1/2}f_0}^{ij} \partial_{v_i, v_j}) f_0, & \text{if } -1 \leq t < 0, \end{cases} \\ &= \begin{cases} -\partial_{v_i} \sigma_G^{ij} \partial_{v_j} f + a_g \cdot \nabla_v f + K_1 f + J_g f, & \text{if } t \geq 0 \\ -f_{0t} + \partial_{v_i} \sigma_{\mu+\mu^{1/2}f_0}^{ij} \partial_{v_j} f_0, & \text{if } -1 \leq t < 0, \end{cases} \end{aligned} \tag{7.1}$$

where σ_G is defined as in (1.12) with $G = \mu + \mu^{1/2}g$ and

$$\begin{aligned} K_1 f &= -\mu^{-1/2} \partial_i \left\{ \mu \left[\phi^{ij} * \left\{ \mu^{1/2} [\partial_j f + v_j f] \right\} \right] \right\} \\ &= 2v_i \mu \left[\phi^{ij} * \left\{ \mu^{1/2} [\partial_j f + v_j f] \right\} \right] - \mu^{1/2} \left[\partial_i \phi^{ij} * \left\{ \mu^{1/2} [\partial_j f + v_j f] \right\} \right], \end{aligned} \quad (7.2)$$

and

$$J_g f = -v \cdot \sigma v f - \partial_i \left\{ \phi^{ij} * [\mu^{1/2} \partial_j g] \right\} f + \left\{ \phi^{ij} * [v_i \mu^{1/2} \partial_j g] \right\} f. \quad (7.3)$$

Lemma 7.1. *For every $\beta > 0$ and $p > 3$,*

$$\|K_1 f\|_{L^p(n \leq |v| \leq n+1)} \leq \frac{C_{p,\beta}}{n^\beta} (\|f\|_{L^p} + \|D_v f\|_{L^p}), \quad (7.4)$$

where K_1 is defined as in (7.2)

Proof. Since K_1 is defined as in (7.2), It is enough to show that

$$\left\| \mu(v) \int_{\mathbb{R}^3} |v - v'|^\vartheta \mu(v') h(v') dv' \right\|_{L^p} \leq \|h\|_{L^p}.$$

By the Hölder inequality,

$$\begin{aligned} & \int_{(0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3} \mu^p(v) \left(\int_{\mathbb{R}^3} |v - v'|^\vartheta \mu(v') h(v') dv' \right)^p dv dx dt \\ & \leq \int_{(0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3} \mu^p(v) \left(\int_{\mathbb{R}^3} |v - v'|^{\vartheta p'} \mu^{p'}(v') dv' \right)^{p/p'} \left(\int_{\mathbb{R}^3} h^p(v') dv' \right) dv dx dt \\ & \leq \int_{\mathbb{R}^3} \mu^p(v) (1 + |v|)^{\vartheta p} dv \int_{(0,\infty) \times \mathbb{T}^3} \left(\int_{\mathbb{R}^3} h^p(v') dv' \right) dx dt \\ & = C_p \|h\|_{L^p}^p. \end{aligned}$$

□

Clearly, we have

$$\|J_g f\|_{L^p(n \leq |v| \leq n+1)} \leq (C + \|g\|_\infty) (\|f\|_{L^p(n \leq |v| \leq n+1)}) \leq C n^{-\beta} (\|(1 + |v|)^\beta f\|_{L^p(n \leq |v| \leq n+1)}). \quad (7.5)$$

Theorem 7.2 (Theorem 1.5 in [23]). *Let Ω be a bounded open set in \mathbb{R}^7 and let f be a strong solution in Ω to the equation*

$$\sum_{i,j=1}^3 \sigma^{ij}(t, x, v) \partial_{v_i, v_j} f + Y f = h,$$

where $Y = -\partial_t - v \cdot \nabla_x$. Suppose that σ is uniformly elliptic,

$$\|\sigma^{ij}\|_{C^\alpha(\Omega)} \leq C, \quad (7.6)$$

and $f, h \in L^p$.

Then $\partial_{v_i, v_j} f \in L_{loc}^p$, $Y f \in L_{loc}^p$ and for every open set $\Omega' \subset \subset \Omega$ there exists a positive constant c_1 depending only on $p, \Omega', \Omega, \alpha, C$ and elliptic constant of σ such that

$$\begin{aligned} \|\partial_{v_i, v_j} f\|_{L^p(\Omega')} &\leq c_1 (\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)}), \\ \|Y f\|_{L^p(\Omega')} &\leq c_1 (\|f\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)}). \end{aligned}$$

Remark 7.3. Especially, from [5]

$$c_1 = c_2(\lambda_1, \lambda_2)c_3(\text{dist}(\Omega', \Omega), \alpha, C, p),$$

where λ_1, λ_2 are the smallest and the largest eigenvalue respectively. More precisely, there exist $C > 0$ and $\alpha' > 0$ such that

$$c_2(\lambda_1, \lambda_2) := \max_{|x|^2 + |v|^2 + |t|^2 = 1} |\Gamma_{v_i, v_j}^0(x, v, t)| \leq C(\lambda_1^{-1} + \lambda_2)^{\alpha'},$$

where $\Gamma^0(\zeta^{-1} \circ z)$ is a fundamental solution of

$$\sum_{i,j=1}^3 \sigma^{ij}(\tau, \xi, \nu) \partial_{v_i, v_j} f + Yf = 0 \quad (7.7)$$

and \circ is a Lie group operation corresponding to (7.7) for some $C > 0$.

Remark 7.4. Since $f(t, \cdot, v)$ is a periodic function on \mathbb{T}^3 , we extend it to a periodic function on $(3\mathbb{T})^3$. Note that

$$\|f\|_{L^p(3\mathbb{T}^3)}^p = 27\|f\|_{L^p(\mathbb{T}^3)}^p.$$

Define

$$\|f\|_{S^p(\Omega)} := \|f\|_{L^p(\Omega)} + \|D_v f\|_{L^p(\Omega)} + \|D_{vv} f\|_{L^p(\Omega)} + \|Yf\|_{L^p(\Omega)},$$

where $Y = -\partial_t - v \cdot \nabla_x$.

Lemma 7.5. *Assume (2.4). Let f be a weak solution of (1.6), (1.7), and (1.25) in the sense of Definition 4.1. Suppose that g satisfies $\|g\|_{C^\alpha((0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq C'$ for some $0 < \alpha < 1$ and $C > 0$. Then there exist $\vartheta > 0$, $p > 3$, $C_{\vartheta, \alpha, C', p}$ such that*

$$\|f\|_{S^p((0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq C(\|f_0\|_{\infty, \vartheta} + \|D_v f_0\|_{\infty, \vartheta} + \|f_0\|_{\infty, \vartheta}). \quad (7.8)$$

Proof. Since $\|g\|_{C^\alpha((0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq C$, σ_G satisfies (7.6). Apply (7.1) to Theorem 7.2 with $\Omega' = \Omega'_n$ and $\Omega = \Omega_n$, where

$$\Omega' = \Omega'_n := \{t \geq 0, x \in \mathbb{T}^3, n \leq |v| \leq n+1\},$$

$$\Omega = \Omega_n := \{t \geq -1, x \in 3\mathbb{T}^3, n-1/2 \leq |v| \leq n+3/2\}.$$

Let σ_G , K_1 , and J_g are defined as in (1.12), (7.2), (7.3). Then, we have

$$\begin{aligned} \|f\|_{S^p(\Omega'_n)}^p &\leq (c_1)^p \left(\|\partial_{v_i} \sigma_G^{ij} \partial_{v_j} f\|_{L^p(\Omega'_n)}^p + \|a_g \cdot \nabla_v f\|_{L^p(\Omega'_n)}^p + \|K_1 f + J_g f\|_{L^p(\Omega'_n)}^p \right. \\ &\quad \left. + \|(v \cdot \nabla_x - \sigma_G^{ij} \partial_{v_i, v_j}) f_0\|_{L^p((-1,0) \times 3\mathbb{T}^3 \times B(0; n-1/2, n+3/2))}^p \right), \end{aligned}$$

where $B(z; r_1, r_2) := B(z; r_2) \setminus B(z; r_1)$. By Lemma 2.3 and Remark 7.3,

$$c_2(\lambda_1, \lambda_2) = Cn^a$$

for some $a > 0$ and

$$c_3(\text{dist}(\Omega', \Omega), \alpha, C, p) = C_{\alpha, C, p}.$$

Therefore,

$$c_1 = C_{\alpha, C, p} n^a$$

for some $a > 0$. Let

$$\tilde{\Omega}'_n = \{t \geq 0, x \in \mathbb{T}^3, n-1/2 \leq |v| \leq n+3/2\}.$$

Then by Remark 7.4,

$$\begin{aligned} \|f\|_{S^p(\tilde{\Omega}'_n)}^p &\leq (C_{\alpha,C,p}n^a)^p \left(\|\partial_{v_i}\sigma_G^{ij}\partial_{v_j}f\|_{L^p(\tilde{\Omega}'_n)}^p + \|a_g \cdot \nabla_v f\|_{L^p(\tilde{\Omega}'_n)}^p + \|K_1f + J_gf\|_{L^p(\tilde{\Omega}'_n)}^p \right. \\ &\quad \left. + \|(v \cdot \nabla_x - \sigma_G^{ij}\partial_{v_i,v_j})f_0\|_{L^p((-1,0)\times\mathbb{T}^3\times B(0;n-1/2,n+3/2))}^p \right), \end{aligned} \quad (7.9)$$

where $\tilde{\Omega}'_n = \{t \geq 0, x \in \mathbb{T}^3, n-1/2 \leq |v| \leq n+3/2\}$. Note that by the standard interpolation, we have

$$\|D_v f\|_{L^p(\tilde{\Omega}'_n)}^p \leq \varepsilon' \|D_{vv}f\|_{L^p(\tilde{\Omega}'_n)}^p + \frac{C}{\varepsilon'} \|f\|_{L^p(\tilde{\Omega}'_n)}^p. \quad (7.10)$$

Let $\beta > 2a + 4$ and $\varepsilon' = \varepsilon_0$ which can be determined later. Then by (7.4), (7.5), and (7.10), we have

$$\begin{aligned} &\sum (C_{\alpha,C,p}n^a)^p \|K_1f + J_gf\|_{L^p(\tilde{\Omega}'_n)}^p \\ &\leq \sum (C_{\beta,\alpha,C,p})^p n^{p(a-\beta)} \left(\|f\|_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}^p + \|D_v f\|_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}^p + \|(1+|v|)^\beta f\|_{L^p(\tilde{\Omega}'_n)}^p \right) \\ &\leq (C_{\beta,\alpha,C,p})^p \left(\|(1+|v|)^\beta f\|_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}^p + \|D_v f\|_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}^p \right) \\ &\leq (C_{\beta,\alpha,C,p,\varepsilon_0})^p \|(1+|v|)^\beta f\|_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}^p + \varepsilon_0 \|D_{vv}f\|_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}^p. \end{aligned} \quad (7.11)$$

for some ε_0 . Similarly,

$$\begin{aligned} &\sum (C_{\alpha,C,p}n^a)^p \|(v \cdot \nabla_x - \sigma_G^{ij}\partial_{v_i,v_j})f_0(x,v)\|_{L^p((-1,0)\times\mathbb{T}^3\times B(0;n-1/2,n+3/2))}^p \\ &\leq \sum (C_{\alpha,C,p})^p n^{p(a-\beta)} \|(1+|v|)^\beta (v \cdot \nabla_x - \sigma_G^{ij}\partial_{v_i,v_j})f_0(x,v)\|_{L^p(\mathbb{T}^3\times B(0;n-1/2,n+3/2))}^p \\ &\leq (C_{\alpha,C,p})^p \|(v \cdot \nabla_x - \sigma_G^{ij}\partial_{v_i,v_j})f_0\|_{p,\beta}^p. \end{aligned} \quad (7.12)$$

Choose $\varepsilon' = \varepsilon n^{-p(a+2)}$ small enough. Since $\|\partial_{v_i}\sigma_G^{ij}\|_\infty < C$, $\|a_g\|_\infty \leq C$, we have

$$\begin{aligned} &\sum (C_{\alpha,C,p}n^a)^p \left(\|\partial_{v_i}\sigma_G^{ij}\partial_{v_j}f\|_{L^p(\tilde{\Omega}'_n)}^p + \|a_g \cdot \nabla_v f\|_{L^p(\tilde{\Omega}'_n)}^p \right) \\ &\leq \sum (C_{\alpha,C,p})^p \left(\varepsilon n^{-2p} \|D_{vv}f\|_{L^p(\tilde{\Omega}'_n)}^p + \varepsilon^{-1} n^{2p(a+1)} \|f\|_{L^p(\tilde{\Omega}'_n)}^p \right) \\ &\leq \sum (C_{\alpha,C,p})^p \left(\varepsilon n^{-2p} \|D_{vv}f\|_{L^p(\tilde{\Omega}'_n)}^p + n^{p(2a+2-\beta)} \|(1+|v|)^\beta f\|_{L^p(\tilde{\Omega}'_n)}^p \right) \\ &\leq (C_{\beta,\alpha,C,p})^p \left(\varepsilon \|D_{vv}f\|_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}^p + C_\varepsilon \|(1+|v|)^\beta f\|_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}^p \right). \end{aligned} \quad (7.13)$$

Combining (7.9) - (7.13) and absorbing $\|D_{vv}f\|$ term on RHS to the LHS, we have

$$\begin{aligned} &\|f\|_{S^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}^p \\ &\leq (C_{\beta,\alpha,C,p})^p \left(\|(1+|v|)^\beta f\|_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}^p + \|(v \cdot \nabla_x - \sigma_F^{ij}\partial_{v_i,v_j})f_0\|_{p,\beta}^p \right) \\ &\leq (C_{\beta,\alpha,C,p})^p \left(\|(1+|v|)^\beta f\|_{L^p((0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3)}^p + \|f_0\|_{p,\beta}^p + \|D_v f_0\|_{p,\beta-2}^p \right). \end{aligned}$$

Finally, by Lemma 2.4, Lemma 5.10, the standard interpolation, and Proposition 2.1, we have (7.8). \square

Here we introduce some regularity results in S^p norm.

Theorem 7.6 (Theorem 2.1 in [23]). *Let $f \in S^p(\mathbb{R}^7)$, with $1 < p < \infty$.*

- (1) *If $2p > 14$ and $p < 14$, then $f \in C^\gamma(\mathbb{R}^7)$, with $\gamma = \frac{2p-14}{p}$;*
- (2) *if $p > 14$, then $\partial_{v_i} f \in C^\delta(\mathbb{R}^7)$, with $\delta = \frac{p-14}{p}$.*

Define $\| (t, x, v) \| = \rho$, where ρ is a unique positive solution to the equation

$$\frac{t^2}{\rho^4} + \frac{|x|^2}{\rho^6} + \frac{|v|^2}{\rho^2} = 1$$

and

$$(\tau, \xi, \nu)^{-1} \circ (t, x, v) = (t - \tau, x - \xi + (t - \tau)\nu, v - \nu).$$

Now we can deduce the following lemma.

Lemma 7.7. *Assume (2.4). Let f be a weak solution of (1.6), (1.7), and (1.25) in the sense of Definition 4.1. Suppose that g satisfies $\|g\|_{C^\alpha((0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq C$ for some $0 < \alpha < 1$ and $C > 0$.*

If $2p > 14$ then, letting $\alpha_1 = \min \left\{ 1, \frac{2p-14}{p} \right\}$, there exist $\vartheta > 0$ and $C = C_{\vartheta, \alpha, C, p}$ such that

$$\frac{|f(t, x, v) - f(\tau, \xi, \nu)|}{\| (\tau, \xi, \nu)^{-1} \circ (t, x, v) \|_{\alpha_1}} \leq C (\|f_0 t\|_{\infty, \vartheta} + \|D_v f_0\|_{\infty, \vartheta} + \|f_0\|_{\infty, \vartheta}).$$

for every $(t, x, v), (\tau, \xi, \nu) \in (0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3$, $(t, x, v) \neq (\tau, \xi, \nu)$.

If $p > 14$ then, letting $\alpha_2 = \frac{p-14}{p}$, there exist $\vartheta > 0$ and $C = C_{\vartheta, \alpha, C, p}$ such that

$$\frac{|\partial_{v_i} f(t, x, v) - \partial_{v_i} f(\tau, \xi, \nu)|}{\| (\tau, \xi, \nu)^{-1} \circ (t, x, v) \|_{\alpha_2}^\beta} \leq C (\|f_0 t\|_{\infty, \vartheta} + \|D_v f_0\|_{\infty, \vartheta} + \|f_0\|_{\infty, \vartheta}).$$

for every $(t, x, v), (\tau, \xi, \nu) \in (0, \infty) \times \mathbb{T}^3 \times \mathbb{R}^3$, $(t, x, v) \neq (\tau, \xi, \nu)$.

Proof. It immediately follows from Lemma 7.5 and Theorem 7.6. \square

Remark 7.8. If $\tau = t$ and $\xi = x$, then $\| (\tau, \xi, \nu)^{-1} \circ (t, x, v) \| = |v - \nu|$.

By Remark 7.8 and Lemma 7.7, we have

Lemma 7.9. *Assume (2.4). Let f be a weak solution of (1.6), (1.7), and (1.25) in the sense of Definition 4.1. Suppose that g satisfies $\|g\|_{C^\alpha((0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq C$ for some $0 < \alpha < 1$ and $C > 0$. Let $p = 14$, then there exist $\vartheta > 0$ and $C = C_{\vartheta, \alpha, C, p}$ such that*

$$\|D_v f\|_{L^\infty((0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3)} C (\|f_0 t\|_{\infty, \vartheta} + \|D_v f_0\|_{\infty, \vartheta} + \|f_0\|_{\infty, \vartheta}).$$

8. PROOF OF THEOREM 1.3

In this section we will use an iteration argument to prove the existence and the uniqueness of the weak solution of (1.5) - (1.7) in the sense of Definition 1.2. We first construct the function sequence as follows. Define $f^{(0)}(t, x, v) := f_0(x, v)$. Since $f_0(x, v)$ satisfies (1.31), by Lemma 4.2, we can define $f^{(1)}$ as a solution of (1.6), (1.7), and (1.25) with $g = f^{(0)}$. Moreover, by Theorem 1.5, $f^{(1)}$ satisfies the assumption in Lemma 2.4. Thus we can also define $f^{(2)}$ as a solution of (1.6), (1.7), and (1.25) with $g = f^{(1)}$. Inductively we can define a function sequence $f^{(n)}$ for $n \geq 0$.

Lemma 8.1. *There exist $C, \vartheta_0 > 0$, and $0 < \varepsilon_0 \ll 1$ such that if f_0 satisfies*

$$\|f_0\|_{\infty, \vartheta} \leq \varepsilon_0,$$

then

$$\sup_{n \in \mathbb{N}, t \geq 0} \|f^{(n)}(t)\|_{\infty, \vartheta} \leq C \|f_0\|_{\infty, \vartheta + \vartheta_0}.$$

Sketch of proof. It is clear by applying Theorem 1.5 to $f^{(n)}$ inductively on n . \square

Lemma 8.2. *Let f_0 be a given function satisfying (1.31). Let f_1 and f_2 be weak solutions of (1.6), (1.7), and (1.25) in the sense of Definition 4.1 with $g = g_1, g = g_2$ respectively. Suppose that g_1 and g_2 are uniformly Hölder continuous functions satisfying (2.4). Then we have*

$$\begin{aligned} & \frac{1}{2} \|(f_1 - f_2)(t)\|_{2, \bar{\vartheta}}^2 + \left(\frac{1}{2} - C\varepsilon \right) \int_0^t \|(f_1 - f_2)(s)\|_{\sigma, \bar{\vartheta}}^2 ds \\ & \leq \int_0^t C(\|f_2\|_{\infty} + \|\nabla_v f_2\|_{\infty}) \min \left\{ \|(g_1 - g_2)(s)\|_{2, \bar{\vartheta}}^2, \|(g_1 - g_2)(s)\|_{\sigma, \bar{\vartheta}}^2 \right\} ds + C \int_0^t \|(f_1 - f_2)\|_{2, \bar{\vartheta}}^2 ds, \end{aligned} \quad (8.1)$$

where $\bar{\vartheta} < 0$ is a constant defined as in Theorem 2.8. Therefore by the Gronwall inequality for every $t_0 > 0$,

$$\sup_{t \in (0, t_0)} \frac{1}{2} \|(f_1 - f_2)(t)\|_{2, \bar{\vartheta}} \leq e^{Ct_0} C_\varepsilon t_0 \sup_{s \in (0, t_0)} \|(g_1 - g_2)(s)\|_{2, \bar{\vartheta}}. \quad (8.2)$$

Proof. Note that f_1, f_2, g_1 , and g_2 satisfy

$$\partial_t(f_1 - f_2) + v \cdot \nabla_x(f_1 - f_2) + L(f_1 - f_2) = \Gamma(g_1, f_1 - f_2) + \Gamma(g_1 - g_2, f_2).$$

Multiplying the above equation by $w^{2\bar{\vartheta}}(f_1 - f_2)$ and integrating both sides of the resulting equation yields

$$\begin{aligned} & \frac{1}{2} \|(f_1 - f_2)(t)\|_{2, \bar{\vartheta}}^2 + \int_0^t \left(w^{2\bar{\vartheta}} L(f_1 - f_2)(s), (f_1 - f_2)(s) \right) ds \\ & = \int_0^t \left(w^{2\bar{\vartheta}} \Gamma(g_1(s), (f_1 - f_2)(s)), (f_1 - f_2)(s) \right) ds + \int_0^t \left(w^{2\bar{\vartheta}} \Gamma((g_1 - g_2)(s), f_2(s)), (f_1 - f_2)(s) \right) ds \\ & = \int_0^t (I) + (II) ds. \end{aligned} \quad (8.3)$$

By Lemma 2.7, we have

$$\left(w^{2\vartheta_0} L(f_1 - f_2)(s), (f_1 - f_2)(s) \right) \geq \frac{1}{2} \|(f_1 - f_2)(s)\|_{\sigma, \bar{\vartheta}}^2 - C \|(f_1 - f_2)(s)\|_{2, \bar{\vartheta}}^2.$$

Since g_1 satisfies (2.4), by Theorem 2.8, we have

$$\begin{aligned} (I) &\leq C \|g_1(s)\|_\infty \|(f_1 - f_2)(s)\|_{\sigma, \bar{\vartheta}}^2 \\ &\leq C\varepsilon \|(f_1 - f_2)(s)\|_{\sigma, \bar{\vartheta}}^2. \end{aligned}$$

Since we want to control (II) in terms of $\|g_1 - g_2\|_{2, \bar{\vartheta}}$, $\|g_1 - g_2\|_{\sigma, \bar{\vartheta}}$, and $\|(f_1 - f_2)(s)\|_{\sigma, \bar{\vartheta}}$, we have to show that

$$\|\nabla_v f_2\|_\infty < \infty.$$

Since g_2 is uniformly Hölder continuous by Lemma 7.9, $D_v f_2$ is uniformly bounded. Therefore by Theorem 2.8, Theorem 1.5, Lemma 7.9, and Young's inequality, we have

$$\begin{aligned} (II) &\leq C (\|f_2\|_\infty + \|\nabla_v f_2\|_\infty) \min\{\|(g_1 - g_2)(s)\|_{2, \bar{\vartheta}}, \|(g_1 - g_2)(s)\|_{\sigma, \bar{\vartheta}}\} \|(f_1 - f_2)(s)\|_{\sigma, \bar{\vartheta}} \\ &\leq C_\varepsilon (\|f_2\|_\infty + \|\nabla_v f_2\|_\infty) \left(\min\left\{\|(g_1 - g_2)(s)\|_{2, \bar{\vartheta}}^2, \|(g_1 - g_2)(s)\|_{\sigma, \bar{\vartheta}}^2\right\} + C\varepsilon \|(f_1 - f_2)(s)\|_{\sigma, \bar{\vartheta}}^2 \right). \end{aligned}$$

Then we have

$$\begin{aligned} &\frac{1}{2} \|(f_1 - f_2)(t)\|_{2, \bar{\vartheta}}^2 + \frac{1}{2} \int_0^t \|(f_1 - f_2)(s)\|_{\sigma, \bar{\vartheta}}^2 ds - C \int_0^t \|(f_1 - f_2)(s)\|_{2, \bar{\vartheta}}^2 ds \\ &\leq \int_0^t C\varepsilon \|(f_1 - f_2)(s)\|_{\sigma, \bar{\vartheta}}^2 ds + C_\varepsilon (\|f_2\|_\infty + \|\nabla_v f_2\|_\infty) \min\{\|(g_1 - g_2)(s)\|_{2, \vartheta_0}^2, \|(g_1 - g_2)(s)\|_{\sigma, \vartheta_0}^2\} ds. \end{aligned}$$

Therefore we have (8.1). \square

Proof of 1.3. Proof of (1)

- **Existence**

Let t_0 be a given positive constant and $\bar{\vartheta} < 0$ be a constant defined as in Theorem 2.8. Since f_0 satisfies (1.31), by Theorem (1.6), $f^{(n)}$ is Hölder continuous uniformly in n . Therefore, by (8.1) we have,

$$\begin{aligned} &\|(f^{(n+1)} - f^{(n)})(t)\|_{2, \vartheta_0}^2 \\ &\leq C' \int_0^t \|(f^{(n)} - f^{(n-1)})(s)\|_{2, \vartheta_0}^2 ds + C \int_0^t \|(f^{(n+1)} - f^{(n)})(s)\|_{2, \vartheta_0}^2 ds. \end{aligned}$$

for some C and C' . Then we will show that

$$\|(f^{(n)} - f^{(n-1)})(t)\|_{2, \vartheta_0}^2 \leq \frac{e^{Cnt_0} (C't)^n}{n!} \quad (8.4)$$

for every $t \in (0, t_0)$ and $n \geq 1$ by the induction on n . Suppose that (8.4) holds for $n = k$, then

$$C' \int_0^t \|(f^{(k)} - f^{(k-1)})(s)\|_{2, \vartheta_0}^2 ds \leq \frac{e^{Ckt_0} (C't)^{k+1}}{(k+1)!}$$

for $t \in (0, t_0)$. Therefore, we have

$$\|(f^{(k+1)} - f^{(k)})(t)\|_{2, \vartheta_0}^2 \leq \frac{e^{Ckt_0} (C't)^{k+1}}{(k+1)!} + C \int_0^t \|(f^{(k+1)} - f^{(k)})(s)\|_{2, \vartheta_0}^2 ds$$

for every $t \in (0, t_0)$. Then by the Gronwall inequality, we have

$$\|(f^{(k+1)} - f^{(k)})(t)\|_{2, \vartheta_0}^2 \leq \frac{e^{Ckt_0} (C't)^{k+1}}{(k+1)!} e^{Ct} \leq \frac{e^{C(k+1)t_0} (C't)^{k+1}}{(k+1)!}$$

for every $t \in (0, t_0)$. Thus we have (8.4) for every $n \in \mathbb{N}$. Moreover, we have

$$\lim_{N \rightarrow \infty} \sum_{n > N} \sup_{0 \leq t \leq t_0} \|(f^{(n)} - f^{(n-1)})(t)\|_{2, \vartheta_0} = 0.$$

Thus $f^{(n)}$ is a Cauchy sequence in $L^2([0, t_0] \times \mathbb{T}^3 \times \mathbb{R}^3, w^{\vartheta_0} dt dx dv)$. Let $f = \lim_{n \rightarrow \infty} f^{(n)}$. Then by (8.2), f is a weak solution of (1.5) - (1.7) in the sense of Definition 1.2.

• **Uniqueness**

Suppose that f and g are weak solutions of (1.5) - (1.7) in the sense of Definition 1.2. Then by (8.1), we have

$$\frac{1}{2} \|(f - g)(t)\|_{2, \vartheta_0}^2 + \left(\frac{1}{2} - C\varepsilon\right) \int_0^t \|(f - g)(s)\|_{\sigma, \vartheta_0}^2 ds \leq C \int_0^t \|(f - g)(s)\|_{2, \vartheta_0}^2 ds.$$

Since $C\varepsilon' < 1/4$, we have

$$\frac{1}{2} \|(f - g)(t)\|_{2, \vartheta_0}^2 \leq C \int_0^t \|(f - g)(s)\|_{2, \vartheta_0}^2 ds.$$

Therefore, by the Gronwall inequality, we have

$$\|(f - g)(t)\|_{2, \vartheta_0}^2 = 0$$

for every $t \in (0, t_0)$.

Since t_0 is arbitrary, we conclude that the weak solution of (1.5) - (1.7) in the sense of Definition 1.2 uniquely exists globally in time.

Proof of (3) We can apply f to Theorem 1.4, Theorem 1.5, Theorem 1.6, and Lemma 7.9. Then we have (1.32) - (1.36).

Proof of (2) Let $F = \mu + \sqrt{\mu} + f$, where f is the weak solution of (1.5) - (1.7) in the sense of Definition 1.2. Consider

$$\partial_t F + v \cdot \nabla_x F = Q(F, F) = \sigma_F^{ij} \partial_{v_i v_j} F + 8\pi F^2. \quad (8.5)$$

Similar to Definition 1.2, we can define a weak solution to (8.5) and we can easily check that F is a weak solution to (8.5). Since f satisfies (1.34), by Lemma 2.4, σ_F is a non-negative definite matrix. Therefore, in a similar manner to Section 3, we can obtain a weak minimum principle for (8.5). Thus, if $F(0) \geq 0$, then $F(t) \geq 0$. □

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